Embedding $\mathcal{PT}$-symmetric BEC subsystems into closed hermitian systems

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 Proposal by Klaiman et al. [PRL 101, 080402 (2008)] for the realization of a real $\mathcal{PT}$ symmetric quantum system:

**BEC in a double well with coherent in- and outcoupling of atoms**

Nonlinear Gross-Pitaevskii equation for Bose-Einstein condensates:

$$\left[-\frac{d^2}{dx^2} + V(x) - g|\psi(x)|^2\right] \psi(x) = \mu \psi(x).$$

The particle gain and loss can be described by a complex potential, e.g.

$$V(x) = \frac{1}{4} x^2 + V_0 e^{-\sigma x^2} + i \Gamma x e^{-\rho x^2}$$

with $V(x) = V^*(-x)$. 

![Complex potential diagram](image-url)
BEC in a $\mathcal{PT}$ symmetric double well with in- and outcoupling of particles:

The $\mathcal{PT}$ symmetric and the symmetry breaking states are excellently described by a two-dimensional matrix model [E.-M. Graefe, JPA 45, 444015 (2012)]:

\[
M = \begin{pmatrix}
-g|\psi_1|^2 - i\Gamma & \nu \\
\nu & -g|\psi_2|^2 + i\Gamma \\
\end{pmatrix}, \quad \mu = \begin{cases} 
-\frac{g}{2} \pm \sqrt{\nu^2 - \Gamma^2}, \\
-g \pm \Gamma \sqrt{\frac{4\nu^2}{g^2 + 4\Gamma^2} - 1}.
\end{cases}
\]
Introduction
Bose-Einstein condensate in a $\mathcal{P}\mathcal{T}$ symmetric double well

Problem:

How can the in- and outflux of particles be realized experimentally?

Proposed solutions:

- Coupling approach with atomic beams

- Time-dependent hermitian four-well or multiwell potentials

- Embedding $\mathcal{P}\mathcal{T}$-symmetric subsystems into closed hermitian systems
Coupling approach with atomic beams

Idea:

- Wave function $\psi_1$ e.g. in a double-$\delta$ potential
- Two propagating waves $\psi_2$ and $\psi_3$ as source and drain

Gross-Pitaevskii equation for the total system:

\[
\mu \psi_1 = \left[ -\Delta - g |\psi_1|^2 + V (\delta(x - a) + \delta(x + a)) \right] \psi_1 \\
+ \eta \psi_2 \delta(x - a) + \tilde{\eta} \psi_3 \delta(x + a),
\]

\[
\mu \psi_2 = \left[ -\Delta - g |\psi_2|^2 + V \delta(x - a) \right] \psi_2 + \eta \psi_1 \delta(x - a),
\]

\[
\mu \psi_3 = \left[ -\Delta - g |\psi_3|^2 + V \delta(x + a) \right] \psi_3 + \tilde{\eta} \psi_1 \delta(x + a).
\]
Coupling approach with atomic beams

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$$\mu \psi_2 = \left[ -\Delta - g |\psi_2|^2 + V \delta(x-a) \right] \psi_2 + \eta \psi_1 \delta(x-a),$$

$$\mu \psi_3 = \left[ -\Delta - g |\psi_3|^2 + V \delta(x+a) \right] \psi_3 + \tilde{\eta} \psi_1 \delta(x+a).$$

Desired behavior: $\eta \psi_2(x_0) = i\Gamma \psi_1(x_0)$, $\tilde{\eta} \psi_3(x_0) = -i\Gamma \psi_1(x_0)$

This enforces:

$$i\Gamma = \frac{\eta |\psi_2(x_0)|}{|\psi_1(x_0)|} e^{i(\varphi_2 - \varphi_1)}, \quad -i\Gamma = \frac{\tilde{\eta} |\psi_3(x_0)|}{|\psi_1(x_0)|} e^{i(\varphi_3 - \varphi_1)}$$

Phase conditions: $\varphi_2 = \varphi_1 + \frac{\pi}{2}$, $\varphi_3 = \varphi_1 - \frac{\pi}{2}$

$\rightarrow$ Can be fulfilled!
Coupling approach with atomic beams

Squared moduli of the stationary ground state and the two propagating waves
Time-dependent hermitian four-well potential

\[ H^{(2)} = \begin{pmatrix} i\Gamma & -J \\ -J & -i\Gamma \end{pmatrix}, \quad H^{(4)}(t) = \begin{pmatrix} E_0(t) & -J_{01}(t) & 0 & 0 \\ -J_{01}(t) & 0 & -J_{12} & 0 \\ 0 & -J_{12} & 0 & -J_{23}(t) \\ 0 & 0 & -J_{23}(t) & E_3(t) \end{pmatrix} \]
Time-dependent hermitian four-well potential

- Observable quantities: \( n_k(t) = |\psi_k(t)|^2 \)
- Dynamics of \( H^{(2)} \):
  \[
  \partial_t n_1 = -j_{12} + 2\Gamma n_1, \quad \partial_t n_2 = j_{12} - 2\Gamma n_2 \\
  \partial_t j_{12} = 2J^2(n_1 - n_2)
  \]
- \( H^{(4)} \) leads to:
  \[
  \partial_t n_1 = j_{01} - j_{12}, \quad \partial_t n_2 = j_{12} - j_{23} \\
  \partial_t j_{12} = 2J^2_{12}(n_2 - n_1) + J_{12}(J_{23}C_{13} - J_{01}C_{02})
  \]
  with \( j_{kl} = iJ_{kl}(\psi_k\psi_l^* - \psi_k^*\psi_l) \) and \( C_{kl} = \psi_k\psi_l^* + \psi_k^*\psi_l \).

Conditions for \( H^{(4)} \) to correctly describe \( H^{(2)} \)
Time-dependent hermitian four-well potential

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Conditions for $H^{(4)}$ to correctly describe $H^{(2)}$

\[
  j_{01} = 2\Gamma n_1,
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Time-dependent hermitian four-well potential

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- Dynamics of \( H^{(2)} \):
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  with \( j_{kl} = iJ_{kl}(\psi_k\psi^*_l - \psi^*_k\psi_l) \) and \( C_{kl} = \psi_k\psi^*_l + \psi^*_k\psi_l \).

Conditions for \( H^{(4)} \) to correctly describe \( H^{(2)} \)

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  j_{01} = 2\Gamma n_1 \quad , \quad j_{23} = 2\Gamma n_2 \quad ,
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Time-dependent hermitian four-well potential

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Conditions for \( H^{(4)} \) to correctly describe \( H^{(2)} \)

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  j_{01} = 2\Gamma n_1 , \quad j_{23} = 2\Gamma n_2 , \quad J_{01}C_{02} = J_{23}C_{12}
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Time-dependent hermitian four-well potential

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- Dynamics of \( H^{(2)} \):
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- \( H^{(4)} \) leads to:
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**Conditions for \( H^{(4)} \) to correctly describe \( H^{(2)} \)**

\[
\begin{align*}
  j_{01} &= 2\Gamma n_1, \\
  j_{23} &= 2\Gamma n_2, \\
  J_{01}C_{02} &= J_{23}C_{12}
\end{align*}
\]

- To fulfill these conditions the potential must be adjusted time-dependently such that \( E_0, E_3, J_{01}, \) and \( J_{23} \) assume the required values.
Creation of a “quasi-stationary” $\mathcal{PT}$-symmetric state in the two middle wells with $n_1(t=0) = n_2(t=0) = 0.5$, $\Gamma/J_{12} = 0.5$

Embedding $\mathcal{PT}$-symmetry into closed hermitian systems

Combination of two double well subsystems into a closed hermitian system:

Subsystem A

Subsystem B

Three variants of the system:
1. Four-dimensional matrix model
2. Model with $\delta$ potentials and spatial resolution of the wave function
3. Model with smooth double well potentials

For simplicity:
Normalization $||\psi_{A,B}|| = 1$ of the wave function in both subsystems.
Ansatz for a four-dimensional hermitian matrix model:

\[
M = \begin{pmatrix}
-g|\psi_{A,1}|^2 & v & -i\gamma & 0 \\
v & -g|\psi_{A,2}|^2 & 0 & +i\gamma \\
+i\gamma & 0 & -g|\psi_{B,1}|^2 & v \\
0 & -i\gamma & v & -g|\psi_{B,2}|^2 \\
\end{pmatrix}
\]

with the wave function

\[
\psi = \begin{pmatrix}
\psi_{A,1} \\
\psi_{A,2} \\
\psi_{B,1} \\
\psi_{B,2} \\
\end{pmatrix} = \begin{pmatrix}
\cos \theta_A e^{+i\varphi_A} \\
\sin \theta_A e^{-i\varphi_A} \\
\cos \theta_B e^{+i\varphi_B+i\varphi_{rel}} \\
\sin \theta_B e^{-i\varphi_B+i\varphi_{rel}} \\
\end{pmatrix}.
\]

States with an without \(\mathcal{PT}\)-symmetry in the two subsystems can be described as solutions of the extended equation

\[
M\psi = \begin{pmatrix}
M_A & M_C \\
M_C^\dagger & M_B \\
\end{pmatrix} \begin{pmatrix}
\psi_A \\
\psi_B \\
\end{pmatrix} = \begin{pmatrix}
\mu_A \psi_A \\
\mu_B \psi_B \\
\end{pmatrix}
\]

with independent eigenvalues \(\mu_{A,B} \in \mathbb{C}\) for both subsystems.
Four-dimensional matrix model

Analytical solutions for $\phi_{\text{rel}} = 0$

- $\mathcal{PT}$-symmetric solutions

With $\theta = \frac{\pi}{4}$, i.e. $\psi = \frac{1}{\sqrt{2}} \left( e^{i\phi}, e^{-i\phi}, e^{-i\phi}, e^{i\phi} \right)$ we obtain the two equations

$$-\frac{g}{2} e^{i\phi} + v e^{-i\phi} - i\gamma e^{-i\phi} = \mu e^{i\phi} \Rightarrow (v - i\gamma) e^{-2i\phi} = \mu + \frac{g}{2},$$

$$-\frac{g}{2} e^{-i\phi} + v e^{i\phi} + i\gamma e^{i\phi} = \mu e^{-i\phi} \Rightarrow (v + i\gamma) e^{2i\phi} = \mu + \frac{g}{2}. $$

By elimination of $\mu + \frac{g}{2}$ we obtain

$$e^{2i\phi} = \pm \sqrt{\frac{v - i\gamma}{v + i\gamma}}$$

and the chemical potential

$$\mu = -\frac{g}{2} \pm \sqrt{v^2 + \gamma^2}.$$

Four-dimensional matrix model
Analytical solutions for $\varphi_{\text{rel}} = 0$

- $\mathcal{PT}$-broken solutions

With $\theta_A = \theta_B = \theta$ and $\varphi_A = \varphi_B = \varphi$ in the ansatz for $\psi$ we obtain

\[-g \cos^3 \theta e^{i\varphi} + v \sin \theta e^{-i\varphi} - i \gamma \cos \theta e^{-i\varphi} = \mu \cos \theta e^{i\varphi}\]
\[-g \sin^3 \theta e^{-i\varphi} + v \cos \theta e^{i\varphi} + i \gamma \sin \theta e^{i\varphi} = \mu \sin \theta e^{-i\varphi}\]

Elimination of $\mu$ yields

\[\sin 2\theta = -\frac{2v}{g} \cos 2\varphi = -\frac{v}{\gamma} \tan 2\varphi,\]

and finally the quasi palindromic polynomial (for $x \equiv e^{2i\varphi}$)

\[x^4 - 2Ax^3 + 2x^2 + 2Ax + 1 = 0 \quad \text{with} \quad A = -i g/2 \gamma.\]

The chemical potential finally reads (for $\gamma > \gamma_c = v^2 \sqrt{4/g^2 - 1}$)

\[\mu = -\frac{g}{2} \left(2 \mp \sqrt{P + \frac{\gamma^2}{v^2} P^2 - P}\right) \quad \text{with} \quad P = \frac{1}{2} \pm \sqrt{\frac{g^2 + 16\gamma^2}{2g}}.\]
Effective two-dimensional system

Wave functions which fulfil the condition $\psi_{A,i} = -i\psi_{B,i} \in \mathbb{R}$ for $i = 1, 2$ lead to decoupled equations for $\psi_A$ and $\psi_B$ and result in the effective two-dimensional model

$$
\begin{pmatrix}
-g|\psi_1|^2 - \gamma & v \\
 v & -g|\psi_2|^2 + \gamma
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
= \mu
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix},
$$

which can be solved with the ansatz $\psi = (\cos \theta, \sin \theta)$. The equations

$$
-g \cos^3 \theta - \gamma \cos \theta + v \sin \theta = \mu \cos \theta
$$
$$
-g \sin^3 \theta + \gamma \sin \theta + v \cos \theta = \mu \cos \theta
$$

yield

$$gy^4 + 4(\gamma + iv)y^3 + 4(-\gamma + iv)y - g = 0 \quad \text{with} \quad y \equiv e^{2i\theta}.$$ 

For given root $y$ of the quartic polynomial (two or four solutions with real $\theta$) the chemical potential $\mu$ can be calculated.

Note: Solutions with $|e^{2i\theta}| \neq 1$ or $|e^{2i\varphi}| \neq 1$ are analytic continuations.
Four-dimensional matrix model
Analytical solutions for $\varphi_{\text{rel}} = 0$

$s$: $\mathcal{PT}$-symmetric states; $a$: $\mathcal{PT}$-broken states; $r$: states of the effective 2d system

$a)
\begin{align*}
g &= 1.5 \\
\text{Re } \mu &= 1 \\
\text{Im } \mu &= 0
\end{align*}$

$b)
\begin{align*}
g &= 3.5 \\
\text{Re } \mu &= -2 \\
\text{Im } \mu &= 0
\end{align*}$
Four-dimensional matrix model

Currents

Currents in the matrix model

\[
\begin{align*}
    j_{A,1\to A,2} &= -v \text{Im} \left( \frac{\psi_{A,1}}{\psi_{A,2}} \right) |\psi_{A,2}|, \\
    j_{A,2\to B,2} &= +\gamma \text{Re} \left( \frac{\psi_{A,2}}{\psi_{B,2}} \right) |\psi_{B,2}|, \\
    j_{B,2\to B,1} &= -v \text{Im} \left( \frac{\psi_{B,2}}{\psi_{B,1}} \right) |\psi_{B,1}|, \\
    j_{B,1\to A,1} &= +\gamma \text{Re} \left( \frac{\psi_{B,1}}{\psi_{A,1}} \right) |\psi_{A,1}|.
\end{align*}
\]

Analytical results

- \(\mathcal{PT}\)-symmetric states (s): \( j = -v \sqrt{\frac{\gamma}{2}} \text{Im} \sqrt{\frac{v-\gamma}{v+i\gamma}} = \sqrt{\frac{v\gamma}{2(v^2+\gamma^2)}} \)
- \(\mathcal{PT}\)-broken states (a): More complicated (all currents differ)
- Effective two-dimensional system (r): \( j = 0 \)
Currents in the four-dimensional matrix model (with $v = 1$) for the $\mathcal{PT}$-symmetric states ($s$), the $\mathcal{PT}$-broken states ($a$), and the states of the effective two-dimensional system ($r$):
Extended models

Gross-Pitaevskii equation for two coupled modes with extended wave functions:

\[
\begin{align*}
-\frac{d^2}{dx^2} - g|\psi_A|^2 + V_{\text{trap}} & \psi_A + iV_{\text{coup}} \psi_B = \mu_A \psi_A, \\
-\frac{d^2}{dx^2} - g|\psi_B|^2 + V_{\text{trap}} & \psi_B - iV_{\text{coup}} \psi_A = \mu_B \psi_B. 
\end{align*}
\]

- delta function potentials:

\[V_{\text{trap}}(x) = V_0^D[\delta(x - b) + \delta(x + b)] ; \quad V_{\text{coup}}(x) = \gamma[\delta(x - b) - \delta(x + b)].\]

- Smooth potentials:

\[V_{\text{trap}}(x) = \frac{1}{4}x^2 + V_0^G e^{-\sigma x^2} ; \quad V_{\text{coup}}(x) = \gamma x e^{-\rho x^2} .\]
delta function potentials

Wave functions

\[
\psi_A, \ |\psi_i|^2, \ \text{Re} \psi_i, \ \text{Im} \psi_i
\]

- A) ground state
- B) excited state
- C) \(P\overline{T}\)-broken state

\[
x, x
\]
Smooth potentials
Wave functions

\[ \psi_A \quad \text{ground state} \]

\[ \psi_B \quad \text{excited state} \]

|\psi_i|^2, \text{Re } \psi_i, \text{Im } \psi_i
Comparisons: delta function potentials and matrix model

Chemical potential

![Graph showing comparisons between delta function potentials and matrix model with varying parameters g and \( \varphi_{\text{rel}} \).](graph.png)
Comparisons: smooth potentials and matrix model

Chemical potential

![Graph showing smooth potentials and matrix model comparisons with different values of $g$. The graph includes plots for $g = 0.2$, $g = 3.0$, and $g = 0.2, \varphi_{rel} = 0.03$. The plots are labeled as a), b), and c) respectively. Each plot shows the real and imaginary parts of the chemical potential against the parameter $\gamma$. The labels include $s_1$, $s_2$, $a_1$, $a_2$, $B_P$, and $B_C$. The matrix model is represented by dashed lines, while the Gauss model is represented by solid lines.]
Conclusion

- $\mathcal{PT}$-symmetric Bose-Einstein condensates can be realized by
  - a coupling approach with atomic beams,
  - time-dependent hermitian four-well or multiwell potentials,
  - embedding $\mathcal{PT}$-symmetric subsystems into closed hermitian systems.

- A four-dimensional matrix model has been solved analytically.
- The analytical results agree qualitatively with numerical simulations of systems with extended wave functions and smooth or delta potentials.

Outlook

- Experimental realization of the coupling mechanism.

References