

# The unification of Mathematics *via* Topos Theory

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Workshop on Constructive Topology  
Palermo, 06 September 2010

# Toposes as unifying spaces in Mathematics

In this lecture, whenever I use the word 'topos', I really mean 'Grothendieck topos'.

Recall that a **Grothendieck topos** can be seen as:

- a **generalized space**
- a **mathematical universe**
- a **geometric theory modulo 'Morita-equivalence'** where 'Morita-equivalence' is the equivalence relation which identifies two (geometric) theories precisely when they have equivalent categories of models in any Grothendieck topos  $\mathcal{E}$ , naturally in  $\mathcal{E}$ .

In this talk, we present a new view of toposes as **unifying spaces** which can serve as **bridges** for transferring information, ideas and results between distinct mathematical theories. This approach, first introduced in my Ph.D. thesis, has already generated ramifications into distinct mathematical fields and points towards a realization of Topos Theory as a **unifying theory of Mathematics**.

# Some examples from my research

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- **Model Theory** (a topos-theoretic interpretation of Fraïssé's construction in Model Theory)
- **Algebra** (an application of De Morgan's law to the theory of fields - jointly with P. T. Johnstone)
- **Topology** (a topological interpretation of the notions of Boolean and De Morgan algebras)
- **Proof Theory** (an equivalence between the traditional proof system of geometric logic and a categorical system based on the notion of Grothendieck topology)
- **Definability** (applications of universal models to definability)

These are just a few examples selected from my Ph.D. work; as I see it, the interest of them especially lies in the fact that they demonstrate the technical usefulness and centrality of the philosophy '*Toposes as bridges*' described below: without much effort, **one can generate an infinite number of new theorems by applying these methodologies!**

# Geometric theories

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## Definition

- A **geometric formula** over a signature  $\Sigma$  is any formula (with a finite number of free variables) built from atomic formulae over  $\Sigma$  by only using finitary conjunctions, infinitary disjunctions and existential quantifications.
- A **geometric theory** over a signature  $\Sigma$  is any theory whose axioms are of the form  $(\phi \vdash_{\vec{x}} \psi)$ , where  $\phi$  and  $\psi$  are geometric formulae over  $\Sigma$  and  $\vec{x}$  is a context suitable for both of them.

## Fact

*Most of the theories naturally arising in Mathematics are geometric; and if a finitary first-order theory is not geometric, we can always associate to it a finitary geometric theory over a larger signature (the so-called **Morleyization** of the theory) with essentially the same models in the category **Set** of sets.*

# The notion of classifying topos

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## Definition

Let  $\mathbb{T}$  be a geometric theory over a given signature. A **classifying topos** of  $\mathbb{T}$  is a Grothendieck topos  $\mathbf{Set}[\mathbb{T}]$  such that for any Grothendieck topos  $\mathcal{E}$  we have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathcal{E})$$

natural in  $\mathcal{E}$ .

## Theorem

*Every geometric theory (over a given signature) has a classifying topos. Conversely, every Grothendieck topos arises as the classifying topos of some geometric theory.*

The classifying topos of a geometric theory  $\mathbb{T}$  can always be constructed canonically from the theory by means of a **syntactic construction**, namely as the topos of sheaves  $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, \mathcal{J}_{\mathbb{T}})$  on the geometric **syntactic category**  $\mathcal{C}_{\mathbb{T}}$  of  $\mathbb{T}$  with respect to the **syntactic topology**  $\mathcal{J}_{\mathbb{T}}$  on it (i.e. the canonical Grothendieck topology on  $\mathcal{C}_{\mathbb{T}}$ ).

# Toposes as bridges I

## Definition

- By a **site of definition** of a Grothendieck topos  $\mathcal{E}$ , we mean a site  $(\mathcal{C}, \mathcal{J})$  such that  $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, \mathcal{J})$ .
- We shall say that two geometric theories are **Morita-equivalent** if they have equivalent categories of models in every Grothendieck topos  $\mathcal{E}$ , naturally in  $\mathcal{E}$ .

Note that 'to be Morita-equivalent to each other' defines an **equivalence relation** of the collection of all geometric theories.

- Two geometric theories are Morita-equivalent if and only if they are biinterpretable in each other (in a generalized sense).
- On the other hand, the notion of Morita-equivalence is a 'semantical' one, and we can expect most of the categorical equivalences between categories of models of geometric theories in **Set** to 'lift' to Morita-equivalences.

# Toposes as bridges II

- Two geometric theories have equivalent classifying toposes if and only if they are Morita-equivalent to each other.
- Hence, a topos can be seen as a *canonical representative* of equivalence classes of geometric theories modulo Morita-equivalence. So, we can think of a topos as embodying the ‘common features’ of mathematical theories which are Morita-equivalent to each other.
- The essential features of Morita-equivalences are all ‘hidden’ inside Grothendieck toposes, and can be revealed by using their different sites of definition.
  - *Conceptually*, a property of geometric theories which is stable under Morita-equivalence *is* a property of their classifying topos.
  - *Technically*, considered the richness and flexibility of topos-theoretic methods, we can expect these properties to be expressible as **topos-theoretic invariants** (in the sense of properties of toposes written in the language of Topos Theory).

# Toposes as bridges III

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- The underlying intuition behind this is that a given mathematical property can manifest itself in several different forms in the context of mathematical theories which have a common 'semantical core' but a different linguistic presentation.
- The remarkable fact is that if the property is formulated as a topos-theoretic invariant on some topos then the expression of it in terms of the different theories classified by the topos is determined to a great extent by the technical relationship between the topos and the different sites of definition for it.
- Indeed, the fact that different mathematical theories have equivalent classifying toposes translates into the existence of **different sites** of definition for **one topos**.
- Topos-theoretic invariants can then be used to **transfer** properties from one theory to another.



# Toposes as bridges IV

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- This idea is technically feasible because the relationship between a site  $(\mathcal{C}, \mathcal{J})$  and the topos  $\mathbf{Sh}(\mathcal{C}, \mathcal{J})$  which it ‘generates’ is often **very natural**, enabling us to easily transfer invariants across different sites.
- A topos thus acts as a ‘**bridge**’ which allows the transfer of information and results between theories which are Morita-equivalent to each other.
- Moreover, the **level of generality** represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories. Indeed, as shown in my thesis, important topos-theoretic invariants considered on the classifying topos  $\mathbf{Set}[\mathbb{T}]$  of a geometric theory  $\mathbb{T}$  translate into interesting logical (i.e. syntactic or semantic) properties of  $\mathbb{T}$ .

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- A topos thus acts as a **'bridge'** which allows the transfer of information and results between theories which are Morita-equivalent to each other.
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- The ‘**working mathematician**’ could very well attempt to formulate his or her properties of interest in terms of topos-theoretic invariants, and derive equivalent versions of them by using alternative sites.
- Also, whenever one discovers a duality or an equivalence of mathematical theories, one should try to interpret it in terms of a Morita-equivalence, calculate the classifying topos of the two theories and apply topos-theoretic methods to extract new information about it.
- There is an strong element of **automatism** in these techniques; by means of them, one can generate new mathematical results without really making any creative effort: indeed, in most cases one can just readily apply the general characterizations connecting properties of sites and topos-theoretic invariants to the particular case of interest.
- On the other hand, the range of applicability of these methods is boundless within Mathematics, by the very **generality** of the notion of topos.

# Toposes as bridges V

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- On the other hand, the range of applicability of these methods is boundless within Mathematics, by the very **generality** of the notion of topos.

# Topos-theoretic invariants

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## Definition

By a **topos-theoretic invariant** we mean a property of (or a construction involving) toposes which is stable under categorical equivalence.

Examples of topos-theoretic invariants include:

- to be a **classifying topos** for a geometric theory.
- to be a **Boolean** (resp. **De Morgan**) topos.
- to be an **atomic** topos.
- to be **equivalent to a presheaf topos**.
- to be a **connected** (resp. **locally connected**, **compact**) topos.
- to be a **subtopos** (in the sense of geometric inclusion) of a given topos.
- to have **enough points**.
- to be **two-valued**.

Of course, there are a great many others, and one can always introduce new ones!

Sometimes, in order to obtain more specific results, it is convenient to consider **invariants of objects** of toposes rather than 'global' invariants of toposes.

# Subtoposes

## Definition

A **subtopos** of a topos  $\mathcal{E}$  is a geometric inclusion of the form  $\mathbf{sh}_j(\mathcal{E}) \hookrightarrow \mathcal{E}$  for a local operator  $j$  on  $\mathcal{E}$ .

## Fact

- A subtopos of a topos  $\mathcal{E}$  can be thought of as an equivalence class of geometric inclusions with codomain  $\mathcal{E}$ ; hence, the notion of subtopos is a **topos-theoretic invariant**.
- If  $\mathcal{E}$  is the topos  $\mathbf{Sh}(\mathcal{C}, J)$  of sheaves on a site  $(\mathcal{C}, J)$ , the subtoposes of  $\mathcal{E}$  are in bijective correspondence with the Grothendieck topologies  $J'$  on  $\mathcal{C}$  which contain  $J$  (i.e. such that every  $J$ -covering sieve is  $J'$ -covering).

# The duality theorem I

## Definition

- Let  $\mathbb{T}$  be a geometric theory over a signature  $\Sigma$ . A **quotient** of  $\mathbb{T}$  is a geometric theory  $\mathbb{T}'$  over  $\Sigma$  such that every axiom of  $\mathbb{T}$  is provable in  $\mathbb{T}'$ .
- Let  $\mathbb{T}$  and  $\mathbb{T}'$  be geometric theories over a signature  $\Sigma$ . We say that  $\mathbb{T}$  and  $\mathbb{T}'$  are **syntactically equivalent**, and we write  $\mathbb{T} \equiv_{\Sigma} \mathbb{T}'$ , if for every geometric sequent  $\sigma$  over  $\Sigma$ ,  $\sigma$  is provable in  $\mathbb{T}$  if and only if  $\sigma$  is provable in  $\mathbb{T}'$ .

## Theorem

*Let  $\mathbb{T}$  be a geometric theory over a signature  $\Sigma$ . Then the assignment sending a quotient of  $\mathbb{T}$  to its classifying topos defines a bijection between the  $\equiv_{\Sigma}$ -equivalence classes of **quotients** of  $\mathbb{T}$  and the **subtoposes** of the classifying topos  $\mathbf{Set}[\mathbb{T}]$  of  $\mathbb{T}$ .*

# The duality theorem II

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The bijection given by the theorem is natural in the following sense. If  $i_{\mathcal{F}} : \mathcal{F} \hookrightarrow \mathbf{Set}[\mathbb{T}]$  is the subtopos of  $\mathbf{Set}[\mathbb{T}]$  corresponding to a quotient  $\mathbb{T}'$  of  $\mathbb{T}$  via the duality theorem, we have a commutative (up to natural isomorphism) diagram in  $\mathbf{CAT}$  (where  $i$  is the obvious inclusion)

$$\begin{array}{ccc} \mathbb{T}'\text{-mod}(\mathcal{E}) & \xrightarrow{\cong} & \mathbf{Geom}(\mathcal{E}, \mathcal{F}) \\ \downarrow i & & \downarrow i_{\mathcal{F}} \circ - \\ \mathbb{T}\text{-mod}(\mathcal{E}) & \xrightarrow{\cong} & \mathbf{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}]) \end{array}$$

naturally in  $\mathcal{E} \in \mathcal{B}\mathcal{T}\text{op}$ .



# A simple example

In light of the fact that the notion of subtopos is a topos-theoretic invariant, the duality theorem allows us to easily transfer information between quotients of geometric theories classified by the same topos.

For example, consider the following problem. Suppose to have a Morita-equivalence between two geometric theories  $\mathbb{T}$  and  $\mathbb{S}$ .

**Question:** If  $\mathbb{T}'$  is a quotient of  $\mathbb{T}$ , is there a quotient  $\mathbb{S}'$  of  $\mathbb{S}$  such that the given Morita-equivalence restricts to a Morita-equivalence between  $\mathbb{T}'$  and  $\mathbb{S}'$ ?

The duality theorem gives a straight **positive answer** to this question. In fact, **both** quotients of  $\mathbb{T}$  and quotients of  $\mathbb{S}$  correspond bijectively with subtoposes of the classifying topos  $\mathbf{Set}[\mathbb{T}] = \mathbf{Set}[\mathbb{S}]$ .

Note the role of the classifying topos as a 'bridge' between the two theories!

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# Applications

The duality theorem realizes a **unification** between the theory of **elementary toposes** and **geometric logic**. In fact, there are several notions and results in elementary Topos Theory which apply to subtoposes of a given topos; all of these results can then be transferred into the world of Logic by passing through the theory of Grothendieck toposes. Examples include:

- The notion of **Booleanization** (resp. **DeMorganization**) of a geometric theory.
- The logical interpretation of the surjection-inclusion factorization of a geometric morphism.
- The Heyting algebra structure on the collection of (syntactic-equivalence classes of) geometric theories over a given signature.

# The Booleanization of a geometric theory

## Definition

Let  $\mathbb{T}$  be a geometric theory over a signature  $\Sigma$ .

- A geometric formula  $\phi(\vec{x})$  over  $\Sigma$  is said to be **consistent** (with respect to  $\mathbb{T}$ ) if the sequent  $\phi(\vec{x}) \vdash_{\vec{x}} \perp$  is not provable in  $\mathbb{T}$ .
- A geometric formula  $\phi(\vec{x})$  over  $\Sigma$  is said to be **stably consistent** (with respect to  $\mathbb{T}$ ) if  $\chi(\vec{x}) \wedge \phi(\vec{x})$  is consistent for each consistent formula  $\chi(\vec{x})$  in the same context.
- The **Booleanization** of  $\mathbb{T}$  is the theory obtained by adding to the axioms of  $\mathbb{T}$  all the geometric sequents of the form  $\top \vdash_{\vec{x}} \phi(\vec{x})$ , where  $\phi(\vec{x})$  is a geometric formula over  $\Sigma$  which is stably consistent with respect to  $\mathbb{T}$ .

Note that on every elementary topos  $\mathcal{E}$ , we have a local operator  $\neg\neg$  on  $\mathcal{E}$  given by the operation of double pseudocomplementation  $\neg\neg : \Omega \rightarrow \Omega$  in the internal Heyting algebra  $\Omega$ .

## Theorem

*Let  $\mathbb{T}$  be a geometric theory over a signature  $\Sigma$ . Then the Booleanization  $\mathbb{T}'$  of  $\mathbb{T}$  corresponds to the subtopos  $\mathbf{sh}_{\neg\neg}(\mathbf{Set}[\mathbb{T}])$  via the duality theorem; in particular,  $\mathbb{T}'$  is classified by  $\mathbf{sh}_{\neg\neg}(\mathbf{Set}[\mathbb{T}])$ .*

# The DeMorganization of the theory of fields

- A subtopos of an elementary topos  $\mathcal{E}$  is said to be **dense** if its corresponding local operator  $j$  satisfies  $j \leq \neg\neg$ .
- For every elementary topos  $\mathcal{E}$ , there exists a largest dense De Morgan subtopos of  $\mathcal{E}$ , called the **DeMorganization** of  $\mathcal{E}$ .
- The DeMorganization of a geometric theory  $\mathbb{T}$  is the quotient of  $\mathbb{T}$  corresponding via the duality theorem to the DeMorganization of the classifying topos of  $\mathbb{T}$ .

## Theorem

- *The DeMorganization of the (coherent) theory of fields is the geometric theory of fields of finite characteristic, in which every element is algebraic over the prime field.*
- *The Booleanization of the (coherent) theory of fields is the theory of algebraically closed fields of finite characteristic, in which every element is algebraic over the prime field.*

# Theories of presheaf type

## Definition

- A geometric theory  $\mathbb{T}$  over a signature  $\Sigma$  is said to be of **presheaf type** if it is classified by a presheaf type.
- A model  $M$  of a theory of presheaf type  $\mathbb{T}$  in the category **Set** is said to be **finitely presentable** if the functor  $\text{Hom}_{\mathbb{T}\text{-mod}(\mathbf{Set})}(M, -) : \mathbb{T}\text{-mod}(\mathbf{Set}) \rightarrow \mathbf{Set}$  preserves filtered colimits.

The class of theories of presheaf type contains all the finitary algebraic theories and many other significant mathematical theories.

## Fact

*For any theory of presheaf type  $\mathcal{C}$ , we have two different representations of its classifying topos:*

$$[f.p.\mathbb{T}\text{-mod}(\mathbf{Set}), \mathbf{Set}] \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, \mathcal{J}_{\mathbb{T}})$$

where  $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$  is the category of finitely presentable  $\mathbb{T}$ -models.

**Note** that this gives us a perfect opportunity to test the effectiveness of the philosophy ‘toposes as bridges’!

# Topos-theoretic Fraïssé's construction

## Theorem

*Let  $\mathbb{T}$  be a theory of presheaf type such that the category  $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})$  satisfies both amalgamation and joint embedding properties. Then any two countable homogeneous  $\mathbb{T}$ -models in  $\mathbf{Set}$  are isomorphic.*

The quotient of  $\mathbb{T}$  axiomatizing the homogeneous  $\mathbb{T}$ -models is precisely the **Booleanization** of  $\mathbb{T}$ .

The theorem is proved by means of an investigation of topos  $\mathbf{Sh}(f.p.\mathbb{T}\text{-mod}(\mathbf{Set})^{\text{op}}, J_{at})$

- geometrically
- syntactically
- semantically

The main result arises from an **integration** of these three lines of investigation according to the principles 'toposes as bridges' explained above.

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*The unification of Mathematics via Topos Theory*,  
arXiv:math.CT/1006.3930, 2010



O. Caramello.

*The Duality between Grothendieck toposes and Geometric Theories*, Ph.D. thesis  
University of Cambridge, 2009