PRESHEAF MODELS FOR CONSTRUCTIVE SET THEORIES

NICOLA GAMBINO

ABSTRACT. We introduce a new kind of models for constructive set theories based on categories of presheaves. These models are a counterpart of the presheaf models for intuitionistic set theories defined by Dana Scott in the '80s. We also show how presheaf models fit into the framework of Algebraic Set Theory and sketch an application to an independence result.

1. VARIABLE SETS IN FOUNDATIONS AND PRACTICE

Presheaves are of central importance both for the foundations and the practice of mathematics. The notion of a presheaf formalizes well the idea of a variable set, that is relevant in all the areas of mathematics concerned with the study of indexed families of objects [19]. One may then readily see how presheaves are of interest also in foundations: both Cohen's forcing models for classical set theories and Kripke models for intuitionistic logic involve the idea of sets indexed by stages.

Constructive aspects start to emerge when one considers the internal logic of categories of presheaves. This logic, which does not include classical principles such as the law of the excluded middle, provides a useful language to manipulate objects and arrows, and can be used as an alternative to diagrammatic reasoning [25]. Furthermore, it is sufficiently expressive to allow the definitions of complex mathematical constructions. This aspect has led to important developments in the study of elementary toposes [16].

The main purpose of this paper is to show how presheaves can be used to obtain models for constructive set theories [23, 5] analogous to the ones defined by Dana Scott for intuitionistic set theories [26]. In order to do so, we will have to overcome the challenges intrinsic to working with *generalised predicative* formal systems. By a generalised predicative formal system we mean here a system that is proof-theoretically reducible to Martin-Löf dependent type theories with W-types and universes [20, 12]. Generalised predicative systems typically contain axioms allowing generalized forms of inductive definitions [1] instead of proof-theoretically strong axioms such as Power Set.

Our development will focus on categories of classes rather than categories of sets as the starting point to define presheaves, thus assuming the perspective of Algebraic Set Theory [15, 27, 7, 22, 6]. The main reason for this choice is that the properties of categories of sets do not always reflect directly the set-theoretical axioms adopted to define them. There are indeed axioms, such as Replacement, that do not express directly properties of sets, but regard the interaction between sets and classes. In categories of classes we can overcome this problem without loss, since sets can be isolated as special objects, those that are in some sense 'small'.

Date: December 10th, 2004.
Research supported by an EPSRC Postdoctoral Fellowship in Mathematics (GR/R95975/01).

One is then led to consider a notion of *small map* in such a way that the axioms of a set theory correspond directly to the axioms for small maps.

This approach has two main advantages. First, it allows us to give a homogeneous treatment of presheaf models of different set theories. Indeed, one of the initial motivations for the research described here was to investigate whether it was possible to generalise the presheaf models for intuitionistic set theories to constructive ones and present them in a uniform way. Secondly, we can show how Scott's presheaf models fit into the paradigm of Algebraic Set Theory.

The study of presheaf and sheaf categories in the generalised predicative setting was initiated by Ieke Moerdijk and Erik Palmgren in [21, 22]. They introduced the notion of *stratified pseudotopos* as a candidate for the notion of a *predicative topos.* A predicative topos should be a counterpart at the generalised predicative level of the notion of an elementary topos. They supported their axiomatisation by showing how stratified pseudotoposes support the construction of internal presheaves, in the sense that categories of internal presheaves in a stratified pseudotopos are again stratified pseudotoposes. They also proved that a stratified pseudotopos can be used to define models of constructive set theories. The combination of these two facts comes close to exhibiting presheaf models for constructive set theories, but does not quite achieve it. This is because categories of classes arising in constructive set theories do not satisfy the axioms for a stratified pseudotopos. For example, as pointed out in the context of intuitionistic set theories in [27], they fail to be *exact.* and, as we will discuss in Subsection 2, they do not have arbitrary *exponentials.* Even if they satisfied all the axioms for a stratified pseudotopos, however, there would still be the problem of obtaining an explicit description of presheaf models, of developing their study, and of finding applications. All of these aspects will be considered here.

Related work on categories of classes in constructive set theories is also presented in [28]. The approach to the construction of categorical models for constructive set theories taken there is slighly different from the one assumed here, even if both follow the perspective of Algebraic Set Theory. While in [28] categorical models for constructive set theories are defined exploiting the existence of a universal object, here they will be obtained via W-types and the assumption of a universal small map.

Let us conclude these introductory remarks with an overview of the contents of the paper. In Section 2 we isolate axiomatically the structure on a category that is necessary to obtain a categorical model for a constructive set theory. We also discuss how the category of classes arising from a constructive set theory is an example of such a structure. That section also serves as an introduction to Algebraic Set Theory. We then shift our attention to presheaves. Section 3 introduces the basic notions, describes the structure on the category of presheaves that is relevant for our study, and defines presheaf models. The study of these models via the so-called Kripke-Joyal semantics is presented in Section 4. We end the paper in Section 5 by sketching an application to an independence result. To be as self-contained as possible, we included background material on category theory and set theory.

2. Classes and sets

2.1. **Set-theoretic axioms.** Set theories based on intuitionistic logic are formulated to provide an axiomatic basis to support the development of intuitionistic mathematics in set theory. Their axioms will be formulated here in an extension of first-order intuitionistic logic with equality, obtained by adding restricted quantifiers of form $(\forall x \in a)$ and $(\exists x \in a)$ as primitive, and standard axioms for them. The membership relation can then be defined. A formula is said to be *restricted* if it contains only quantifiers that are restricted.

Classes provide a convenient notation to manipulate sets and formulas in mathematical practice. If ϕ is an formula with a free variable x and A is defined by $A =_{\mathrm{def}} \{x \mid \phi\}$, we let $a \in A =_{\mathrm{def}} \phi[a/x]$. Two classes are said to be *extensionally equal* if they have the same elements. Note that every set can be viewed as a class, and that proper classes cannot be considered as elements of other classes. Equality between sets is disciplined by the axiom of Extensionality, stating that two sets are equal if they are extensionally equal as classes.

The basic set existence axioms of Pairing, Union, and Infinity, familiar from classical set theory, simply assert that certain classes are sets. Using these, standard definitions allow us to introduce the forms of classes

$$A \times B\,, \quad A + B\,, \quad \textstyle\sum_{a \in A} B_a\,,$$

that denote binary cartesian products, binary disjoint unions, and indexed disjoint unions, respectively. There is a natural notion of function between classes, that generalises the notion of function between sets. We write $f : A \to B$ to express that f is a function from A to B, and by this we mean that f is a subclass of $A \times B$ that is total and single-valued as a relation from A to B. Two functions are considered equal if they are extensionally equal as classes.

The set-theoretic universe, defined by

$$V =_{\mathrm{def}} \{x \mid x = x\}\,, \tag{1}$$

cannot be asserted to be a set if we wish to avoid Russell's paradox. The interplay between classes and sets is specified further by other axioms. Restricted Separation is the axiom scheme asserting that for each set a and each restricted formula ϕ, the class $\{x \in a \mid \phi\}$ is a set. This weakening of the usual Full Separation axiom scheme is sufficient for many purposes in mathematical practice. In the absence of Full Separation, one needs to distinguish carefully between *subsets* and *subclasses* of a set. When considering two sets, we assume that functions between them are given as subsets, rather than subclasses, of the cartesian product of the domain and codomain.

The axiom of Exponentiation, originally introduced in [23], asserts that the class of functions between any two sets is again a set. Exponentiation is a consequence of Power Set that deserves to be isolated if one wishes to consider set theories that do not include Power Set. Strong Collection is the scheme

$$(\forall x \in a)(\exists y)\phi \to (\exists u)\big((\forall x \in a)(\exists y \in u)\phi \wedge (\forall y \in u)(\exists x \in a)\phi\big)\,,$$

where a is a set and ϕ is an arbitrary formula. A consequence of Strong Collection is Replacement, asserting that for a function $f : A \to B$ between classes, if A is a set then so is its image. Strong Collection also allows us to have generalised forms of inductive definitions of classes [5, Chapter 5]. For instance, if A is a class and $(B_a \mid a \in A)$ is family of sets, we can form the associated class $W_{a \in A} B_a$ of *wellfounded trees*. This is defined as the smallest class X such that if $a \in A$ and $t \in X^{B_a}$ then $(a, t) \in X$. Classes of wellfounded trees are a set-theoretic counterpart of the well-ordering types introduced by Martin-Löf [20].

The axiom of Set Induction is used to view the set-theoretic universe as an inductively-defined class. It asserts that the class V defined in (1) is the smallest class X such that, for all sets p, if $p \subseteq X$ then $p \in X$. Set Induction is relevant in our study since it allows us to see the set-theoretic universe as an initial algebra for an endofunctor on the category of classes, and this suggests a way to formulate a general notion of model for set theories with Set Induction, as we discuss in Subsection 2.3.

In the following, we will focus our attention on the constructive set theory CST whose axioms are Extensionality, Set Induction, Pairing, Union, Infinity, Restricted Separation, Exponentiation and Strong Collection. This is an extension of Myhill's original system [23], obtained by adding Strong Collection, and a subsystem of Aczel's system CZF, obtained by replacing Subset Collection with Exponentiation [5]. Robert Lubarsky has recently proved that Subset Collection is independent of Exponentiation, thus showing that CST is a proper subsystem of CZF [17]. The intuitionistic set theory IZF which is is essentially an intuitionistic counterpart of classical Zermelo-Frankel set theory, is obtained from CST by adding Full Separation and Power Set [9]. Presheaf and sheaf models for IZF have been considered in [26, 8].

2.2. **Categories of classes.** We now associate to the constructive set theory CST a category, called **CST**, and study how the axioms of CST determine the properties of **CST**. The category **CST** is defined as having classes as objects and functions between them as arrows. In the study of the properties of this category, we follow [6].

The category **CST** has an obvious terminal object, given by the singleton class $1 =_{\text{def}} \{\emptyset\}$. Pullbacks in **CST** will now be defined explicitly. Given two arrows $u : X \to A$ and $f : B \to A$, we let $Y =_{\text{def}} \{(b, x) \in B \times X \mid fb = ux\}$. The required pullback diagram is then given by

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "v"'] & X \arrow[d, "u"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

where v and g are the first and second projection, respectively. Sometimes we will write $B \times_A X$ to denote the class Y defined above. The operation of pullback can be thought of as reindexing, or substitution: if one regards an arrow $u : X \to A$ as a family $(X_a \mid a \in A)$, where $X_a =_{\text{def}} \{x \in X \mid ux = a\}$, for $a \in A$, then, in the pullback diagram above, the family $(Y_b \mid b \in B)$ is isomorphic to the family $(X_{fb} \mid b \in B)$, obtained by reindexing the family $(X_a \mid a \in A)$ via the function f. Since **CST** has a terminal object and pullbacks, it has all finite limits [18, Section V.2].

For an arrow $f : B \to A$ we may consider the pullback of f along itself. This determines a diagram of form $B \times_A B \rightrightarrows B$ that is called the *kernel pair* of f. Such a diagram determines an equivalence relation on B according to which $b\,, b' \in B$ are related if and only if $f(b) = f(b')$ holds. The quotient of B under this equivalence relation is isomorphic to the image of f in A, written $\mathrm{Im}(f)$ here, and we thus we have a diagram of form

$$B \times_A B \rightrightarrows B \longrightarrow \mathrm{Im}(f) \tag{2}$$

that is *exact* [15, Appendix B]. The possibility of defining quotients of this kind can be expressed abstractly by saying that we have *coequalizers of kernel pairs*. The arrow $B \to \mathrm{Im}(f)$ is obviously an surjection and it is easy to show that in **CST** every surjection is a *regular epimorphism*, i.e. that it fits into a diagram of the form in (2). Surjections are stable, in the sense that the pullback of a surjection is again a surjection. This discussion indicates that **CST** is a *regular category*, in the sense specified by the next definition.

Definition 2.1. A category $\mathcal{E}$ is *regular* if it has finite limits, coequalizers of kernel pairs, and if regular epimorphisms (i.e. epimorphisms that arise as coequalizers of kernel pairs) are stable under pullback.

The category **CST** has also *disjoint finite coproducts*, given by disjoint unions, and these are stable in the sense that they are preserved by pullbacks. In particular, the empty coproduct is given by $0 =_{\mathrm{def}} \emptyset$. The regular structure of **CST** is sufficient to define an adjunction $\exists_f \dashv \Delta_f$ between the functors

$$\mathit{Sub}\, A \underset{\exists_f}{\overset{\Delta_f}{\rightleftarrows}} \mathit{Sub}\, B$$

for any arrow $f : B \to A$, where we write $\mathit{Sub}\, X$ for the category of subclasses of a class X, with arrows given by inclusions. The functors Δ_f and $\exists_f$ are defined by letting, for $P \subseteq A$ and $Q \subseteq B$

$$\Delta_f(P) =_{\mathrm{def}} \{b \in B \mid fb \in P\}\,, \quad \exists_f(Q) =_{\mathrm{def}} \{a \in A \mid (\exists b \in B_a)\, b \in Q\}\,.$$

This indicates that regular categories have enough structure to interpret a large fragment of first-order intuitionistic logic, which however does not include the universal quantifier [13, Section 4.4]. To interpret it, one requires *dual images*, i.e. the existence of a right adjoint $\forall_f : \mathit{Sub}\, B \to \mathit{Sub}\, A$ to the functor Δ_f, for every $f : B \to A$. These right adjoints exist in **CST** and can be defined by letting, for $Q \subseteq B$

$$\forall_f(Q) =_{\mathrm{def}} \{a \in A \mid (\forall b \in B_a)\, b \in Q\}\,.$$

So far, we have described some of the structure available on categories of classes. As pointed out in [27], categories of classes arising from intuitionistic or constructive set theories generally fail to be *exact*, in the sense that it is not possible to define quotients of arbitrary equivalence relations, but only of equivalence relations whose equivalence classes are sets.

Further structure on the category **CST** arises from the interplay between classes and sets, but does not seem to be directly expressible in terms of universal properties. For example, for two classes A and B, the functions from A to B can be collected into a class only if A is set, since elements of classes are required to be sets. This means that **CST** does not have arbitrary exponentials, but has exponentials of sets. More generally, if we have a set A and a family of classes $(B_a \mid a \in A)$ we can define the class $\prod_{a \in A} B_a$ whose elements are the functions f with domain A such that, for all $a \in A$, $f(a) \in B_a$.

To axiomatize this situation, one may introduce a notion of *small map*. The next definition introduces a natural notion of small map for the category **CST**.

Definition 2.2. An arrow $u : X \to A$ in **CST** is said to be *small* if, for all a in A, the class $X_a =_{\mathrm{def}} \{x \mid ux = a\}$ is a set.

Note that a set is just a class for which the canonical arrow into the singleton set 1 is small. In the next subsection, we will arrive at a definition of a categorical model for CST by isolating a group of axioms that are satisfied by the small maps in the regular category **CST**.

2.3. **Axioms for small maps.** In [15, 22] axioms for small maps were considered in the context of *pretoposes*, but they can be studied in the more general context of regular categories with stable disjoint coproducts. This is indeed the setting that we consider here, and from now on we let $\mathcal{E}$ be a regular category with stable disjoint coproducts. In order to introduce these axioms, it is convenient to recall some basic notions.

For $A \in \mathcal{E}$, one may define $\mathcal{E}/A$, the *slice category over* A, whose objects are arrows $u : X \to A$ of $\mathcal{E}$. For two objects $u : X \to A$, $u' : X' \to A$ of $\mathcal{E}/A$, an arrow in $\mathcal{E}/A$ between them is given by $v : X \to X'$ in $\mathcal{E}$ such that $u = u'\, v$ holds. In the category **CST** it is convenient to represent an object $X \to A$ of $\mathcal{E}/A$ as the family $(X_a \mid a \in A)$. The operation of pullback along $f : B \to A$ determines a functor $\Delta_f : \mathcal{E}/A \to \mathcal{E}/B$. This functor always has a left adjoint $\Sigma_f : \mathcal{E}/B \to \mathcal{E}/A$ defined by composition with f. The action of Σ_f can be described in **CST** using indexed disjoint unions.

For $f : B \to A$ in $\mathcal{E}$, if the pullback functor $\Delta_f : \mathcal{E}/A \to \mathcal{E}/B$ has a right adjoint $\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$, we can define $P_f : \mathcal{E} \to \mathcal{E}$, the *generalised polynomial functor* associated to f, as the composite

$$\mathcal{E} \xrightarrow{\Delta_B} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_A} \mathcal{E}$$

where we made use of the canonical arrows $A : A \to 1$ and $B : B \to 1$.

In the category **CST**, the functor Π_f can be defined if $f : B \to A$ is small. Identifying objects in the slice categories with families of classes, we define the functor Π_f by letting, for a family $(Y_b \mid b \in B)$ indexed by B,

$$\Pi_f(Y_b \mid b \in B) =_{\mathrm{def}} (\textstyle\prod_{b \in B_a} Y_b \mid a \in A) \tag{3}$$

We can then obtain the following explicit description of the generalised polynomial functor associated to an arrow $f : B \to A$. For a class X we have

$$P_f(X) = \textstyle\sum_{a \in A} X^{B_a}\,.$$

By its very definition, the class of wellfounded trees $W_{a \in A} B_a$ is an initial algebra for this functor. This observation leads to define a general notion of wellfounded tree in categories [21]. These are defined as initial algebras for generalised polynomial functors. Note that a natural numbers object can be characterised as an initial algebra for the polynomial functor associated to either of the two canonical arrows $1 \to 1 + 1$. For more on wellfounded trees, see [21, 11].

The next definition isolates axioms for small maps corresponding to the properties of small maps for CST, as we will discuss after the definition. Our axioms are imposed on top of a standard group of axioms, those for a *class of open maps*, that are not recall here. These are axioms (**A1**) – (**A7**) of [15, Section 1.1], and were first formulated in [14]. The axioms presented below are tailored to construct models of CST. Axioms (**S1**), (**S2**) were introduced in [15, Definition 1.1], while (**S3**) was considered in [14, 6]. By a *small object* we mean an object for which the canonical map into the terminal object is small.

Definition 2.3. Let $\mathcal{E}$ be a regular category with stable disjoint coproducts, and $\mathcal{S}$ a family of open maps. We say that $\mathcal{S}$ is a family of CST-*small maps* if the following axioms hold.

(S1): If $f : B \to A$ is in $\mathcal{S}$, the pullback functor $\Delta_f : \mathcal{E}/A \to \mathcal{E}/B$ has a right adjoint $\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$.

(S2): There is a map $\pi : W \to V$ in $\mathcal{S}$ such that for any map $u : X \to A$ in $\mathcal{S}$ there exists a diagram of form

$$\begin{tikzcd} X \arrow[d, "u"'] & Y \arrow[l] \arrow[r] \arrow[d] & W \arrow[d, "\pi"] \\ A & B \arrow[l, two heads, "f"] \arrow[r] & V \end{tikzcd}$$

where $f : B \twoheadrightarrow A$ is an epimorphism and both squares are pullbacks.

(S3): For every $f : B \to A$, the canonical map $B \to B \times_A B$ is in $\mathcal{S}$.

(S4): If $f : B \to A$ is in $\mathcal{S}$, then $\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$ preserves smallness of maps.

(S5): For every $f : B \to A$ in $\mathcal{S}$, the polynomial functor $P_f : \mathcal{E} \to \mathcal{E}$ has an initial algebra, whose underlying object is written W_f here. The natural numbers object is small.

(S6): For every $f : B \to A$ in $\mathcal{S}$ there is an exact diagram

$$\begin{tikzcd} E \arrow[r, shift left] \arrow[r, shift right] & W_f \arrow[r] & V_f \end{tikzcd}$$

where $E \rightarrowtail W_f \times W_f$ is a small subobject such that

$$(\forall (a,t),\, (a',t') \in W_f)\,\Big(\big((a,t),(a',t')\big) \in E \leftrightarrow$$
$$(\forall x \in a')(\exists x' \in a')(tx, t'x') \in E \wedge (\forall x' \in a')(\exists x \in a)(tx, t'x') \in E\Big)\,.$$

holds in the internal logic of $\mathcal{E}$.

We now wish to indicate how a category $\mathcal{E}$ with the properties of Definition 2.3 provides a categorical model for CST. The idea follows essentially from the sets-as-trees interpretation of constructive set theory [2, 3, 4].

First, consider the small map $\pi : W \to V$ of axiom (**S2**) and its associated wellfounded tree W_π. In the category **CST**, a map satisfying axiom (**S2**) can be defined by letting $W =_{\mathrm{def}} \{(x,y) \mid y \in x\}$, $V = \{x \mid x = x\}$, and taking π to be the first projection. Indeed, this map satisfies a stronger property than the one of axiom (**S2**) in that every small map in **CST** can be obtained simply as a pullback of it. The wellfounded tree associated to this family is then defined as the smallest class X such that if a is a set and t is a function with domain a, then (a,t) is in X. Such a pair (a,t) may be thought of as a non-extensional set, given as the family $(tx \mid x \in a\,)$. The non-extensionality means for example that tx and tx' are considered distinct elements of the family even if $tx = tx'$. To obtain extensional sets, it is therefore necessary to take a quotient.

We define V_π to be the quotient of the wellfounded tree W_π under the equivalence relation of axiom (**S6**). In **CST** this equivalence relation, denoted by $\cdot \simeq \cdot$, is such that for (a,t) and (b,s) in W_π, we have

$$(a,t) \simeq (b,s) \leftrightarrow (\forall x \in a)(\exists y \in b)tx \simeq sy \wedge (\forall y \in b)(\exists x \in a)tx \simeq sy\,.$$

The quotient of W_π under this equivalence relation is a class that is isomorphic to the cumulative hierarchy. In an arbitrary category $\mathcal{E}$, the object V_π is then a natural candidate to model CST. The canonical subobject

$$V_\pi \rightarrowtail^{=_{V_\pi}} V_\pi \times V_\pi$$

is used to interpret the equality relationship of the language of CST. The interpretation of the rest of the syntax is done as usual [13, Section 4.5], and restricted quantifiers are interpreted using appropriately small maps [22, Remark 3.8]. We do not need to define the interpretation of the membership relation, as we assumed that this is defined using equality and restricted quantifiers.

Theorem 2.4 (Soundness and completeness)**.**

- *If $\mathcal{E}$ is a regular category with disjoint coproducts and dual images, equipped with a class of* CST*-small maps, the object V_π of $\mathcal{E}$ is such that $(V_\pi, =_{V_\pi})$ is a model of* CST.
- **CST** *is a regular category with disjoint coproducts and dual images, equipped with a family of* CST*-maps.*

Proof. The first part of the theorem follows essentially by Theorem 7.1 of [22]. Definition 2.3 was indeed introduced to isolate the essential elements of the proof necessary to prove the claim.

For the second part of the theorem, one uses the axioms of CST to verify the required conditions. The existence of the structure of a regular category with disjoint coproducts and dual images has already been discussed in Subsection 2.2. The functors required (**S1**) are defined as in (3), and (**S2**) is a consequence of the definition of $\pi : W \to V$ given above. For (**S3**) one should recall that equality is a restricted formula, while for (**S4**) one uses Exponentiation and Restricted Separation. Finally, axioms (**S5**) and (**S6**) hold because we can define wellfounded trees by induction and relations on them by double set-recursion [12], and the class of natural numbers is a set by Infinity. □

It is possible to have an alternative form of intuition about the cumulative hierarchy object V_π defined above. For a small map $f : B \to A$ in $\mathcal{E}$ let us consider the generalised polynomial functor $P_f : \mathcal{E} \to \mathcal{E}$ and recall that in **CST** this can be expressed as

$$P_f(X) = \textstyle\sum_{a \in A} X^{B_a} \, .$$

In the presence of sufficiently strong axioms for quotients, that are valid in the category of classes, one can then follow [22, Section 6] and define suitable quotients of $P_f(X)$ so as to determine a functor $\mathbb{P}_f : \mathcal{E} \to \mathcal{E}$. In the special case of the category **CST**, this is expressed as

$$\mathbb{P}_f(X) = \{ \, p \mid p \subseteq X \, , \; (\exists a \in A)(\exists t \in X^{B_a}) \, p = \mathrm{Im}(t) \, \} \, .$$

In **CST**, the functor $\mathbb{P}_\pi$ associated to the arrow $\pi : W \to V$ defined above is exactly the *power-class operation*, defined by letting $\mathbb{P}(X) =_{\mathrm{def}} \{p \mid p \subseteq X\}$, where X is a class. Note that elements of $\mathbb{P}(X)$ are subsets of X, rather than subclasses, and that generally $\mathbb{P}(X)$ is a class, even when X is a set.

Just as the set-theoretic universe is an initial algebra for $\mathbb{P} : \mathbf{CST} \to \mathbf{CST}$ by the Set Induction axiom, the object V_π defined above is an initial algebra for the functor $\mathbb{P}_\pi : \mathcal{E} \to \mathcal{E}$, as proved in [22]. This point of view will be exploited in the next section, where we define models of CST in categories of presheaves. In order to do so, we recall a property that characterizes the functor $\mathbb{P}_\pi : \mathcal{E} \to \mathcal{E}$ from [15, 27]. We first need a definition.

Definition 2.5. For X, $I \in \mathcal{E}$, we say that a subobject $R \rightarrowtail I \times X$ is an *I-indexed family of small subobjects of X* if $R \rightarrowtail I \times X \to I$ is a small map.

The functor $\mathbb{P}_\pi : \mathcal{E} \to \mathcal{E}$ has the property (**P1**), expressed as in [27].

(P1): For every object X there is an $\mathbb{P}_\pi(X)$-indexed family of small subobjects of X, $\ni_X \rightarrowtail \mathbb{P}_\pi(X) \times X$, such that for all I-indexed families of small subobjects of X, $R \rightarrowtail I \times X$, there exists a unique map $\bar{R} : I \to \mathbb{P}_\pi(X)$ for which there is a pullback diagram of form

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & \ni_X \arrow[d, tail] \\ I \times X \arrow[r, "\bar{R} \times 1_X"'] & \mathbb{P}_\pi(X) \times X \end{tikzcd}$$

It is easy to verify that this property holds in **CST**. For a proof that the property (**P1**) holds in pretoposes, see [15, Section 1.3]. In [27] the property (**P1**) was introduced as part of a simplified axiomatization of categorical models for IZF. It would be of interest to formulate a simple axiom that allows us to derive both the existence of the quotient required by axiom (**S6**) and the definability of the functors $\mathbb{P}_\pi : \mathcal{E} \to \mathcal{E}$.

3. Presheaves

In this section, we work again with the constructive set theory CST and consider the category **CST** of classes arising from it. We will use **CST** to define new categories equipped with a class of small maps satisfying all the axioms of Definition 2.3. One could have considered, more generally, to work internally in an arbitrary category $\mathcal{E}$ that satisfies the axioms of Definition 2.3 rather than with the explicitly defined category **CST**. The approach taken here, however, will be sufficient to show how Dana Scott's presheaf models [26] fit into the framework of Algebraic Set Theory, and has the advantage of keeping the presentation quite simple.

3.1. **Basic definitions.** In this section, $\mathbb{C}$ is a fixed small category, in the sense that we assume that objects and arrows of $\mathbb{C}$ form a set in the constructive set theory CST. Objects and arrows are denoted with $a\,, b\,, c\,, \ldots$, and $f : b \to a\,, g : c \to b\,, h : d \to c\,, \ldots$, respectively. The identity arrow on an object a will be written $1_a : a \to a$ and the composite of f and g as above will be written $f\,g : c \to a$. For a, $b \in \mathbb{C}$, we let $[b, a]$ be the set of arrows from b to a. To help the intuition, one may think of an object a as a stage in a process, and of an arrow $f : b \to a$ in $\mathbb{C}$ as a transition from the stage a to the stage b.

The *opposite* of $\mathbb{C}$ is the category $\mathbb{C}^{\text{op}}$ whose objects are the same of $\mathbb{C}$ and whose arrows are obtained by formally reversing the direction of the arrows of $\mathbb{C}$. A *presheaf* is a functor $\mathbb{C}^{\text{op}} \to$ **CST**. Thus, a presheaf $X : \mathbb{C}^{\text{op}} \to$ **CST** consists of a family of classes $X(a)$, for $a \in \mathbb{C}$, together with a family of functions

$$\begin{array}{ccc} X(a) \times [b, a] & \longrightarrow & X(b) \\ (\ x\ ,\ f\ ) & \longmapsto & x \cdot f \end{array}$$

that satisfy the equations

$$x \cdot 1_a = x\,, \quad (x \cdot f) \cdot g = x \cdot f\, g$$

for all $x \in X(a)$, and $f : b \to a\,,\ g : c \to b$. We may think of a presheaf X as a class varying through stages, and of the members of $X(a)$, for $a \in \mathbb{C}^{\mathrm{op}}$, as the elements of the variable class X at stage a. The function $X(a) \to X(b)$ determined by an arrow $f : b \to a$ can then be imagined as describing the evolution of the variable class X along the transition f.

As usual, we write $\widehat{\mathbb{C}}$ for the category of presheaves and natural transformations between them. If X and Y are presheaves, we say that a Y is a *subpresheaf* of X if, for all $a \in \mathbb{C}$, $Y(a) \subseteq X(a)$ holds. We now give a few examples of presheaves.

(i) For a class X, we define $X : \mathbb{C}^{\mathrm{op}} \to \mathbf{CST}$, the *constant* presheaf associated to X, as the functor mapping every object into X and any arrow into the identity map on X. The constant presheaf determined by 1 is a terminal object in $\widehat{\mathbb{C}}$.
(ii) For $a \in \mathbb{C}^{\mathrm{op}}$, we define $\mathbf{y}_a : \mathbb{C}^{\mathrm{op}} \to \mathbf{CST}$, the presheaf *represented by* a, by letting $\mathbf{y}_a(b) =_{\mathrm{def}} [b, a]$, for $b \in \mathbb{C}$. The actions are defined so that a pair $(f, g) \in \mathbf{y}_a(b) \times [c, b]$ is mapped into $f\, g \in \mathbf{y}_a(c)$.
(iii) Binary products, and disjoint binary coproducts are defined pointwise. For details, see [19].

The next definition shows how the category $\widehat{\mathbb{C}}$ inherits a notion of small map from $\mathbf{CST}$. The definition of small map in $\mathbf{CST}$ was given in Definition 2.2.

Definition 3.1. A natural transformation $\alpha : X \to Y$ is said to be *small* if, for all $a \in \mathbb{C}$, the function between classes $\alpha_a : X(a) \to Y(a)$ is a small map in $\mathbf{CST}$.

To define a presheaf model of CST, i.e. a model of CST whose underlying object is a presheaf, one may prove that $\widehat{\mathbb{C}}$ is a category satisfying all the hypotheses of the first part of Theorem 2.4 by combining the ideas in [15, Section IV§3] and in [21, 22]. We prefer, however, to give an explicit description of the model. This will be achieved by defining explicitly a power-presheaf functor that satisfies axiom (**P1**) and then isolating an initial algebra for it.

3.2. **Presheaf models.**

Definition 3.2. Let $X \in \widehat{\mathbb{C}}$. For $a \in \mathbb{C}$, we say that p is a *presheaf subset of* X *at stage* a if the following hold:

(i) p is a function with domain $\bigcup_{b \in \mathbb{C}} \mathbf{y}_a(b)$,
(ii) for all $b \in \mathbb{C}$ and $f : b \to a$, $p(f)$ is a set, and it holds that $p(f) \subseteq X(b)$,
(iii) for all b, $c \in \mathbb{C}$ and $f : b \to a\,, g : c \to b$, if $x \in p(f)$ then $x \cdot g \in p(f\, g)$.

The next lemma states a useful property of presheaf subsets. Its proof is a direct consequence of Definition 3.2.

Lemma 3.3. *Let* p *be presheaf subset of* X *at stage* a, *for* $X \in \widehat{\mathbb{C}}$ *and* $a \in \mathbb{C}$. *We have* $(\forall x \in X_a)\ x \in p(1_a)$ *if and only if* $(\forall b \in \mathbb{C})(\forall f \in \mathbf{y}_a(b))\ x \cdot f \in p(f)\,.$

We begin to define the power-presheaf functor $\mathbb{P}_\pi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by letting,

$$\mathbb{P}_\pi(X)(a) =_{\mathrm{def}} \{p \mid p \text{ presheaf subset of } X \text{ at stage } a\}\,.$$

for $X \in \widehat{\mathbb{C}}$ and $a \in \mathbb{C}$. To define the action

$$\begin{array}{ccc} \mathbb{P}_\pi(X)(a) \times [b, a] & \longrightarrow & \mathbb{P}_\pi(X)(b) \\ (\ p\ ,\ f\ ) & \longmapsto & p \cdot f \end{array}$$

we let $p \cdot f$ be the function mapping $g \in [c, b]$ into $p(f\,g)$. The functoriality required by the definition of presheaf follows directly. We are almost ready to verify the property of the power-presheaf functor stated in axiom (**P1**). We only need to exhibit, for every presheaf X, a $\mathbb{P}_\pi(X)$-indexed family of small subobjects of X that plays the role of a membership relation. For $X \in \widehat{\mathbb{C}}$ we define $\ni_X$ by letting

$$(\ni_X)(a) =_{\text{def}} \{(p, x) \in \mathbb{P}_\pi(X)(a) \times X(a) \mid x \in p(1_a)\}$$

for $a \in \mathbb{C}$. Observe that $\ni_X$ is a subpresheaf of $\mathbb{P}_\pi(X) \times X$ by Lemma 3.3 and that it is a $\mathbb{P}_\pi(X)$-indexed family of small subobjects of X.

Proposition 3.4. *The functor $\mathbb{P}_\pi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfies the property* (**P1**).

Proof. For an I-indexed family of small subobjects of X, $R \rightarrowtail I \times X$, we need to define a natural transformation $\bar{R} : I \to \mathbb{P}_\pi(X)$. For $a \in \mathbb{C}$, $i \in I(a)$ and $f : b \to a$ we define

$$\bar{R}_a(i)(f) =_{\text{def}} \{y \in X(b) \mid (i \cdot f, y) \in R(b)\}\,,$$

where we assumed that R is a subpresheaf of $I \times X$ for simplicity. This definition determines a function $\bar{R}_a(i)$ that is a presheaf subset of X at stage a, so that $\bar{R}_a(i) \in \mathbb{P}_\pi(X)$. For $a \in \mathbb{C}$, we then have a function $\bar{R}_a : I(a) \to \mathbb{P}_\pi(X)(a)$, which gives us the components of the required natural transformation. □

Dana Scott's definition of a presheaf cumulative hierarchy in [26] is exactly an initial algebra for the power-presheaf functor. The next result recalls its characterization and states that the required presheaf can be defined in CST.

Theorem 3.5. *We can define a presheaf V_π such that, for $a \in \mathbb{C}$, it holds that $s \in V_\pi(a)$ if and only if the following conditions hold:*

- *s is a function with domain $\bigcup_{b \in \mathbb{C}} \mathbf{y}_a(b)$,*
- *for all $f : b \to a$, we have $s(f) \in V_\pi(b)$,*
- *for all $f : b \to a$ and $g : c \to b$, if $t \in s(f)$ then $t \cdot g \in s(f\,g)$,*

where, for $s \in V_\pi(a)$ and $f : b \to a$, we let $s \cdot f$ be the function with domain $\bigcup_{c \in \mathbb{C}} \mathbf{y}_b(c)$ mapping $g : c \to b$ into $s(f\,g)$.

Proof. The claim is a consequence of the possibility of defining classes by general forms of inductive definitions in CST. The details of the appropriate inductive definition are given in [10, Section 6.3], where it is also shown how V_π can be seen as the smallest presheaf that satisfies the requirements above. □

4. Kripke-Joyal semantics

The presheaf V_π can be used to give a direct interpretation of all the axioms of CST using the Kripke-Joyal semantics [19, Section VI.6]. To do so, we need to fix some syntactic conventions. For $a \in \mathbb{C}$, we define the language $\mathcal{L}(a)$ to be the extension of the language of CST with constants for elements of $V_\pi(a)$. As usual, we do not distinguish between elements of $V_\pi(a)$ and the constants added to the language $\mathcal{L}$ and use letters $s, t, r, \ldots$ for them. If ϕ is a formula of $\mathcal{L}(a)$ and $f : b \to a$ we define $\phi \cdot f$ to be the formula obtained from ϕ by leaving unchanged free variables and substituting each constant s appearing in ϕ with the constant $s \cdot f$. Observe that if ϕ is a sentence of $\mathcal{L}(a)$ then $\phi \cdot f$ is a sentence of $\mathcal{L}(b)$. The Kripke-Joyal semantics can then be defined by structural induction as in Table 1. Lemma 4.1 then states one of the expected properties of the semantics.

$$
\begin{aligned}
a \Vdash \bot &=_{\mathrm{def}} \bot \\
a \Vdash s = t &=_{\mathrm{def}} s = t \\
a \Vdash \phi \wedge \psi &=_{\mathrm{def}} (a \Vdash \phi) \wedge (a \Vdash \psi) \\
a \Vdash \phi \vee \psi &=_{\mathrm{def}} (a \Vdash \phi) \vee (a \Vdash \psi) \\
a \Vdash \phi \rightarrow \psi &=_{\mathrm{def}} (\forall b \in \mathbb{C})(\forall f \in \mathbf{y}_a(b))(b \Vdash \phi \cdot f \rightarrow b \Vdash \psi \cdot f) \\
a \Vdash (\exists x \in s)\phi &=_{\mathrm{def}} \big(\exists x \in s(1_a)\big)\, a \Vdash \phi \\
a \Vdash (\forall x \in s)\phi &=_{\mathrm{def}} (\forall b \in \mathbb{C})(\forall f \in \mathbf{y}_a(b))\,(\forall x \in s(f))\, b \Vdash \phi \cdot f \\
a \Vdash (\exists x)\phi &=_{\mathrm{def}} \big(\exists x \in V_\pi(a)\big)\, a \Vdash \phi \\
a \Vdash (\forall x)\phi &=_{\mathrm{def}} (\forall b \in \mathbb{C})(\forall f \in \mathbf{y}_a(b))\big(\forall x \in V_\pi(b)\big)\, b \Vdash \phi \cdot f
\end{aligned}
$$

TABLE 1. Definition of the Kripke-Joyal semantics.

Lemma 4.1 (Monotonicity). *Let $a \in \mathbb{C}$ and ϕ be a sentence of $\mathcal{L}(a)$. If $a \Vdash \phi$ then for all $b \in \mathbb{C}$ and all $f \in \mathbf{y}_a(b)$ it holds that $b \Vdash \phi \cdot f$.*

Proof. The claim follows by structural induction on ϕ . □

To illustrate some of the properties of presheaf models, we investigate in more detail the interpretation of sentences. Let $a \in \mathbb{C}$. We say that a class P of arrows with codomain a is a *sieve* on a if for all $f : b \to a$ and $g : c \to b$ if $f \in P$ then $f\,g \in P$. We say that a sieve on a is a *set-sieve* if it is a set. Let a, $b \in \mathbb{C}$. For a set-sieve p on a and $f : b \to a$, define $p \cdot f =_{\mathrm{def}} \{g \mid f\,g \in p\}$ and observe that $p \cdot f$ is a set-sieve on b. We can then define a presheaf Ω by letting, for $a \in \mathbb{C}$

$$\Omega_a =_{\mathrm{def}} \{p \mid p \text{ set-sieve on } a\}$$

The next definition provides a link between the Kripke-Joyal semantics of sentences and sieves. For a sentence ϕ of $\mathcal{L}(a)$ define

$$[\![\phi]\!] =_{\mathrm{def}} \textstyle\bigcup_{b \in \mathbb{C}} \{f \in [b, a] \mid b \Vdash \phi \cdot f\}\,.$$

Proposition 4.2. *Let a in $\mathbb{C}_0$. Let ϕ be a sentence of $\mathcal{L}(a)$. The class $[\![\phi]\!]$ is a sieve on a, and if ϕ is restricted then $[\![\phi]\!]$ is a set-sieve on a.*

Proof. For the first claim, Lemma 4.1 gives the desired conclusion. For the second, use structural induction on ϕ, observing that the clauses defining the semantics of a restricted formula are themselves restricted. □

Theorem 4.3. *$(V_\pi, =)$ is a model of* CST.

Proof. The claim follows mainly from Theorem 6.17 in [10], apart from the validity of Exponentiation, which is a consequence of Theorem 7.1 and Theorem 9.6 of [22]. □

5. CONCLUSIONS

Since partially ordered sets are special small categories, presheaf models give as a special case extensions for constructive set theories of Kripke models for intuitionistic logic. In [24] Erik Palmgren applied a Kripke model construction to show an independence result for first-order intuitionistic logic. The result regards the notion of *pseudo-order* on a set. A binary relation $\cdot < \cdot$ on a set A is a pseudo-order if the following hold:

(1) $\neg\big((a < b) \wedge (b < a)\big)$, for all $a, b \in A$
(2) $\neg(a < b) \wedge \neg(b < a) \rightarrow a = b$, for all $a, b \in A$
(3) $a < b \rightarrow \big((a < c) \vee (c < a)\big)$, for all $a, b, c \in A$

An example of pseudo-order is given by the strict order on Cauchy or Dedekind reals. Classically, every pseudo-order is a linear order and thus every two elements have a supremum. Palmgren's result shows that this is not the case in intuitionistic logic. For the proof, obtained using a Kripke model, see [24].

Proposition 5.1 (Palmgren)**.** *In intuitionistic first-order logic, the axioms of a pseudo-order do not imply that every two elements of a pseudo-order have a supremum.*

It seems straightforward to generalise this result to an independence result for CST using presheaf models. Presheaf models offer, however, much more generality. One may indeed consider models, relevant for the study of the lambda-calculus [26], in which the category $\mathbb{C}$ is taken to be a monoid. Furthermore, it is possible to generalise presheaf models to sheaf models [15, 22, 10] which give variants of the double-negation translation suitable for constructive set theories. In this paper, we set the ground for these promising applications.

Acknowledgements. I am most grateful to Dana Scott, who made available his notes on presheaf models. I also acknowledge useful discussions with Alex Simpson, Peter Aczel, Steve Awodey, Erik Palmgren, Benno van den Berg, Claire Kouwenhoven-Gentil, and Robert Lubarsky.

References

[1] P. Aczel. An introduction to inductive definitions. In J. Barwise, editor, *Handbook of Mathematical Logic*, chapter 7, pages 739 – 782. North-Holland, 1977.
[2] P. Aczel. The type theoretic interpretation of Constructive Set Theory. In A. MacIntyre nd L. Pacholski and J. Paris, editors, *Logic Colloquium '77*, pages 55 – 66. North-Holland, 1978.
[3] P. Aczel. The type theoretic interpretation of Constructive Set Theory: choice principles. In A. S. Troelstra and D. van Dalen, editors, *The L. E. J. Brouwer Centenary Symposium*, pages 1 – 40. North-Holland, 1982.
[4] P. Aczel. The type theoretic interpretation of Constructive Set Theory: inductive definitions. In R. Barcan Marcus, G.J.W. Dorn, and P. Weinegartner, editors, *Logic, Methodology and Philosophy of Science VII*, pages 17 – 49. North-Holland, 1986.
[5] P. Aczel and M. Rathjen. Notes on Constructive Set Theory. Technical Report 40, Mittag-Leffler Institute, 2001. Available from the web page `http://www.cs.man.ac.uk/~petera/papers.html`.
[6] S. Awodey, C. Butz, T. Streicher, and A. K. Simpson. Relating topos theory and set theory via categories of classes. Draft paper, available from the web page `http://www.andrew.cmu.edu/user/awodey`, 2003.
[7] C. Butz. Bernays-Gödel type theory. *J. Pure Appl. Algebra*, 178(1):1 – 23, 2003.
[8] M. P. Fourman. Sheaf models for set theory. *Journal of Pure and Applied Algebra*, 19:91 – 101, 1980.
[9] H. M. Friedman. The consistency of classical set theory relative to a set theory with intuitionistic logic. *J. of Symbolic Logic*, 38:315–319, 1973.
[10] N. Gambino. *Sheaf interpretations for generalised predicative intuitionistic systems.* PhD thesis, University of Manchester, 2002.

[11] N. Gambino and M. Hyland. Wellfounded trees and dependent polynomial functors. In S. Berardi, M. Coppo, and F. Damiani, editors, *Types for Proofs and Programs*, volume 3085 of *Lecture Notes in Computer Science*. Springer, 2004.

[12] E. R. Griffor and M. Rathjen. The strength of some Martin-Löf type theories. *Arch. for Math. Logic*, 33:347 – 385, 1994.

[13] B. Jacobs. *Categorical Logic and Type Theory*. North-Holland, 1999.

[14] A. Joyal and I. Moerdijk. A completeness theorem for open maps. *Annals of Pure and Applied Logic*, 70:51 – 86, 1994.

[15] A. Joyal and I. Moerdijk. *Algebraic Set Theory*. Cambridge University Press, 1995.

[16] A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. *Mem. Amer. Math. Soc.*, 51(309), 1984.

[17] R. S. Lubarsky. Independence results around Constructive ZF. *Annals of Pure and Applied Logic*, To appear.

[18] S. MacLane. *Categories for the working mathematician*. Springer, 1971.

[19] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic*. Springer, 1992.

[20] P. Martin-Löf. *Intuitionistic Type Theory*. Bibliopolis, 1984.

[21] I. Moerdijk and E. Palmgren. Wellfounded trees in categories. *Ann. of Pure and Appl. Logic*, 104:189 – 218, 2000.

[22] I. Moerdijk and E. Palmgren. Type theories, toposes and Constructive Set Theory: predicative aspects of AST. *Ann. of Pure and Appl. Logic*, 114(1-3):155–201, 2002.

[23] J.R. Myhill. Constructive Set Theory. *J. of Symbolic Logic*, 40(3):347–382, 1975.

[24] E. Palmgren. Constructive completions of ordered sets, groups and fields. Technical Report U.U.D.M. Report 2003:5, Department of Mathematics, University of Uppsala, 2003.

[25] A. M. Pitts. Categorical Logic. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 5. Oxford University Press, 2000.

[26] D. S. Scott. Category-theoretic models for Intuitionistic Set Theory. Manuscript slides of a talk given at Carnagie-Mellon University, 1985.

[27] A. K. Simpson. Elementary axioms for the category of classes. In *14th Annual IEEE Symposium on Logic in Computer Science*, pages 77 – 85. IEEE Press, 1999.

[28] M. Warren. Predicative categories of classes. Master's thesis, Carnagie Mellon University, 2004.

DPMMS, University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, England

E-mail address: N.Gambino@dpmms.cam.ac.uk