Dependent Type Theory

Lecture 4

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References

S. Awodey and M. A. Warren
Homotopy theoretic models of identity types
Math. Proc. Camb. Phil. Soc. 146 (1) 2009, pp. 45-55.

. . . and R. Garner
The identity type weak factorisation system
Theoretical Computer Science 409 (1) 2008, pp. 94-109.

Outline

1. Identity types
2. Weak factorisation systems

Identity types (I)

Formation rule.

$$\frac{A \in \mathsf{Type} \qquad a \in A \qquad b \in A}{\mathsf{Id}_A(a, b) \in \mathsf{Type}}$$

Introduction rule.

$$\frac{a \in A}{\mathsf{r}(a) \in \mathsf{Id}_A(a, a)}$$

Identity types (II)

Elimination rule.

$$\frac{p \in \mathrm{Id}_A(a,b) \qquad (x \in A)\ d(x) \in C(x,x,\mathrm{r}(x))}{\mathrm{J}(a,b,p,d) \in C(a,b,p)}$$

Computation rule.

$$\frac{a \in A \qquad (x \in A)\ d(x) \in C(x,x,\mathrm{r}(x))}{\mathrm{J}(a,a,\mathrm{r}(a),d) = d(a) \in C(a,a,\mathrm{r}(a))}$$

Identity types (III)

Expansion rules.

$$\frac{p \in \mathrm{Id}_A(a, b)}{a = b \in A}$$

$$\frac{p \in \mathrm{Id}_A(a, b)}{p = \mathrm{r}(a) \in \mathrm{Id}_A(a, a)}$$

Semantics of identity types in LCCC

Idea. Let us try to model only the basic rules.

For every $A \in \mathbb{C}$, we want

- $\mathrm{Id}_A \to A \times A$
- $\mathrm{r} : A \to \mathrm{Id}_A$

such that

- $A \to \mathrm{Id}_A \to A \times A$ is the diagonal.
- Every commutative square

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "\mathrm{r}"'] & C \arrow[d] \\
\mathrm{Id}_A \arrow[r, equal] & \mathrm{Id}_A
\end{tikzcd}$$

has a diagonal filler.

Problem

Consider the commutative square

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "r"'] & A \arrow[d] \\ \mathrm{Id}_A \arrow[r, equal] & \mathrm{Id}_A \end{tikzcd}$$

This forces

$$A \cong \mathrm{Id}_A$$

This, in turn, validates the expansion rules. These have been shown to be independent from the other rules by Hofmann and Streicher.

Problems

Conceptual.

The extra assumption of deduction rules for identity types should correspond to additional structure.

Practical.

The assumption of expansion rules in sufficiently rich dependent type theories makes type-checking undecidable (Hofmann).

New approach

Homotopical algebra.

Many naturally-arising mathematical structures satisfy weak forms of the standard algebraic laws.

What is the combinatorics of higher homotopies?

Vast development: operads, weak factorisation systems, Quillen model categories, $(\omega, 1)$-categories, …

Weak factorisation systems (I)

Definition. Let $\mathbb{C}$ be a category. Let $i : A \to B$, $p : C \to D$ in $\mathbb{C}$. We write

$$i \pitchfork p$$

if every diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

has a diagonal filler.

For $\mathcal{A}, \mathcal{B}$, we write $\mathcal{A} \pitchfork \mathcal{B}$ if $i \pitchfork p$ for all $i \in \mathcal{A}$ and $p \in \mathcal{B}$.

For $\mathcal{A}, \mathcal{B}$, we define $\mathcal{A}^{\pitchfork} = \{p \mid \mathcal{A} \pitchfork \{p\}\}$ and ${}^{\pitchfork}\mathcal{B} = \{i \mid \{i\} \pitchfork \mathcal{B}\}$.

Weak factorisation systems (II)

Let $\mathbb{C}$ be a category.

A **weak factorisation system** on $\mathbb{C}$ is a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathbb{C}$ such that:

- every $A \xrightarrow{f} B$ can be factored as $A \xrightarrow{i} X \xrightarrow{p} B$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$,
- $\mathcal{A}^{\pitchfork} = \mathcal{B}$ and $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$.

Example

Let **Gpd** be the category of groupoids. Define:

- $\mathcal{A} =$ functors that are equivalences and injective on objects.
- $\mathcal{B} =$ Grothendieck fibrations.

Factor $F : \mathbb{A} \to \mathbb{B}$ as

$$\begin{tikzcd}
\mathbb{A} \arrow[r] \arrow[dr, bend right, "F"'] & \{(a, b, \beta) \mid a \in \mathbb{A}\,,\ b \in \mathbb{B}\,,\ \beta : F(a) \to b\} \arrow[d] \\
& \mathbb{B}
\end{tikzcd}$$

The syntactic category

The category **Syn**(T) has:

- **Objects.** Contexts $\Gamma, \Delta, \ldots$
- **Maps.** Given

$$\begin{aligned} \Gamma &= (x_0 \in A_0, x_1 \in A_1(x_0), \ldots, x_n \in A_n(x_0, \ldots, x_{n-1})) \\ \Delta &= (y_0 \in B_0, y_1 \in B_1(y_0), \ldots, y_m \in B_m(y_0, \ldots, y_{m-1})) \,, \end{aligned}$$

a map $f : \Gamma \to \Delta$ is a sequence $f = (b_0, \ldots, b_m)$ such that

$$\begin{gathered} (\Gamma)\ b_0 \in B_0 \\ (\Gamma)\ b_1 \in B_1(b_0) \\ \ldots \\ (\Gamma)\ b_m \in B_m(b_0, \ldots, b_{m-1}) \end{gathered}$$

Identities are evident, composition by substitution.

Display maps

Display maps have the form

$$\pi_A : (\Gamma, x \in A) \to (\Gamma)$$

Recall that they are closed under pullback:

$$\begin{tikzcd}
(\Gamma, x \in A) \arrow[r] \arrow[d] & (\Delta, y \in B) \arrow[d] \\
(\Gamma) \arrow[r] & (\Delta)
\end{tikzcd}$$

Let us write $\mathcal{D}$ for the class of display maps.

The identity type weak factorisation system

Let T be any fragment of **ML** with rules for identity types and Π-types.

Theorem. The category **Syn**(T) admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ defined by letting $\mathcal{A} = {}^{\pitchfork}\mathcal{D}$ and $\mathcal{B} = \mathcal{A}^{\pitchfork}$.

Note. The assumption of Π-types can be removed by formulating ‘parametrized rules’ for identity types.

The key lemma

Lemma. Every map f can be factored as $f = pi$, with $i \in \mathcal{A}$ and p a composite of display maps.

The theorem then follows by the Retract Argument.

Idea.

We factor $f : (x \in A) \to (y \in B)$ as

$$(x \in A) \xrightarrow{i} \big(x \in A, y \in B, u \in \mathrm{Id}_B(f(x), y)\big) \xrightarrow{p} (y \in B)$$

where $i = (x, f(x), \mathrm{r}(f(x)))$ and $p = (y)$.

- We have that p is a composite of display maps.
- We need to show $\{i\} \pitchfork \mathcal{D}$.

Example

The identity $(x \in A) \to (x \in A)$ factors

$$(x \in A) \xrightarrow{i} (x \in A, y \in A, u \in \mathrm{Id}_A(x, y)) \xrightarrow{p} (y \in A)$$

Need to fill

$$\begin{tikzcd}
(x \in A) \arrow[r] \arrow[d] & (x \in A, y \in A, u \in \mathrm{Id}_A(x, y), z \in C(x, y, u)) \arrow[d] \\
(x \in A, y \in A, u \in \mathrm{Id}_A(x, y)) \arrow[r, equal] & (x \in A, y \in A, u \in \mathrm{Id}_A(x, y))
\end{tikzcd}$$

Use

$$\frac{(x \in A)\ d(x) \in C(x, x, \mathrm{r}(x))}{(x \in A, y \in A, u \in \mathrm{Id}_A(x, y))\ \mathrm{J}(x, y, u, d) \in C(x, y, u)}$$

The fundamental groupoid of a type

We can define a functor

$$\mathcal{F} : \mathbf{Syn}(T) \to \mathbf{Gpd}$$

such that

- $\mathcal{A}$-maps are sent to injective equivalences,
- $\mathcal{B}$-maps are sent to Grothendieck fibrations.

Idea. Composition and inverses are given by

$$\frac{p \in \mathrm{Id}_A(a, b) \qquad q \in \mathrm{Id}_B(b, c)}{q \circ p \in \mathrm{Id}_A(a, c)} \qquad \frac{p \in \mathrm{Id}_A(a, b)}{p^{-1} \in \mathrm{Id}_A(b, a)}$$

Semantics

Let $(\mathcal{A}, \mathcal{B})$ be a weak factorisation system on a category $\mathbb{C}$.

Idea.

- Interpret $(x \in A, y \in B(x))$ as a $\mathcal{B}$-**map** over $[\![x \in A]\!]$.
- Interpret
$$(x \in A, y \in A, u \in \mathsf{Id}(x, y))$$
by factoring the diagonal map
$$[\![x \in A]\!] \longrightarrow P \longrightarrow [\![x \in A]\!] \times [\![x \in A]\!]$$
and letting
$$[\![x \in A, y \in A, u \in \mathsf{Id}(x, y)]\!] = P$$

Validity

The premisses of the elimination rule give us:

$$\begin{tikzcd}
\llbracket x \in A \rrbracket \arrow[r, "\llbracket d \rrbracket"] \arrow[d] & \llbracket x \in A, y \in A, u \in \mathrm{Id}_A(x,y), z \in C(x,y,u) \rrbracket \arrow[d] \\
\llbracket x \in A, y \in A, u \in \mathrm{Id}_A(x,y) \rrbracket \arrow[r, equal] & \llbracket x \in A, y \in A, u \in \mathrm{Id}_A(x,y) \rrbracket
\end{tikzcd}$$

The interpretation of

$$(x \in A, y \in A, u \in \mathrm{Id}_A(x,y)) \;\; J(x,y,u,d) \in C(x,y,u)$$

is defined to be the diagonal filler.

Further developments

B. van den Berg and R. Garner
Types are weak omega-groupoids
ArXiv:0812.0298
December 1st, 2008

P. L. Lumsdaine
Weak omega-categories from intensional type theory
ArXiv:0812.0409
December 2nd, 2008