Regular and Singular Pulse and Front Solutions and Possible Isochronous Behavior in the Extended-Reduced Ostrovsky Equation: Phase-plane, multi-infinite series and variational formulations

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Abstract

In this paper we employ three recent analytical approaches to investigate several classes of traveling wave solutions of the so-called extended-reduced Ostrovsky Equation (exROE). A recent extension of phase-plane analysis is first employed to show the existence of breaking kink wave solutions and smooth periodic wave (compacton) solutions. Next, smooth traveling waves are derived using a recent technique to derive convergent multi-infinite series solutions for the homoclinic orbits of the traveling-wave equations for the exROE equation. These correspond to pulse solutions respectively of the original PDEs. We perform many numerical tests in different parameter regime to pinpoint real saddle equilibrium points of the corresponding traveling-wave equations, as well as ensure simultaneous convergence and continuity of the multi-infinite series solutions for the homoclinic orbits anchored by these saddle points. Unlike the majority of unaccelerated convergent series, high accuracy is attained with relatively few terms. And finally, variational methods are employed to generate families of both regular and embedded solitary wave solutions for the exROE PDE. The technique for obtaining the embedded solitons incorporates several recent generalizations of the usual variational technique and it is thus topical in itself. One unusual feature of the solitary waves derived here is that we are able to obtain them in analytical form (within the assumed ansatz for the trial functions). Thus, a direct error analysis is performed, showing the accuracy of the resulting solitary waves. Given the importance of solitary wave solutions in wave dynamics and information propagation in nonlinear PDEs, as well as the fact that not much is known about solutions of the family of generalized exROE equations considered here, the results obtained are both new and timely.

Key words: Extended-Reduced Ostrovsky Equation, Traveling Waves, Singular
1 Introduction

In the papers [26], [29], extensive classifications have been made of the periodic, solitary (pulse), and compacton traveling wave solutions of the so-called extended-reduced Ostrovsky equation (exROE), the long-wave limit [28] of the so-called Ostrovsky equation [25] governing nonlinear internal waves in rotating flows. In this paper, we investigate solutions of this equation by three other techniques.

Various analytical methods have been developed to construct solitary waves of physically important nonlinear partial differential equations (NLPDEs), including variational methods, diverse series solution techniques, the extended tanh—function method, Hirota’s method, truncated regular and invariant Painlevé expansions, and various others.

Three of these techniques are applied to the exROE equation in this paper. First, novel phase-plane methods are used to consider singular solutions of the exROE equation, in particular breaking kink or front solutions. We next employ one recently developed technique to construct convergent, multi-infinite, series solutions for regular solitary waves of the exROE equation (or equivalently, homoclinic orbits of its traveling-wave equation). In addition, in an alternative approach, the variational method is employed to construct regular solitary waves of the exROE NLPDE directly, and also attempt to construct embedded solitons of the PDE using several recent extensions of the variational approach.

The remainder of the paper is organized as follows. In Section 2, the traveling wave ODE of the exROE equation is considered. A recently developed technique (see [7], [27]) is employed to construct convergent series solutions for its homoclinic and heteroclinic orbits, corresponding to solitary wave and front (pulse) solutions of the original exROE NLPDE. A Lagrangian for the exROE equation is developed in Section 3. Section 4 then considers the linear spectrum of the exROE equation to isolate the parameter regimes where regular solitary waves exist. A Gaussian ansatz or trial function for these solitary waves is then substituted into the Lagrangian, and its Euler-Lagrange
equations are solved to derive the optimum soliton or ansatz parameters in the usual way (within the functional Gaussian form of the ansatz).

2 Singular Solutions of the Extended-Reduced Ostrovsky equation

Let us consider the following traveling wave equation associated to the extended reduced Ostrovsky equation (exROE):

$$\frac{d}{dz} \left( \phi \frac{d\phi}{dz} \right) + \frac{1}{2} (p + q) \phi^2 + (pc + \beta) \phi = k, \quad (1)$$

where $\phi = \phi(z)$, $p, q$ and $\beta$ are constant coefficients and $k$ is a constant of integration. Recall that the exROE has been derived in [26] from the Hirota-Satsuma-type shallow water wave equation, and has been also discussed in [29]. Equation (1) is equivalent to the following 2-dimensional system:

$$\begin{cases}
\frac{d\phi}{dz} = y, \\
\frac{dy}{dz} = \frac{k - (pc + \beta)\phi - \phi y^2 - \frac{1}{2}(p + q)\phi^2}{\phi^2},
\end{cases} \quad (2)$$

which is the traveling wave system for (1). The system (2) belongs to the following class of system:

$$\begin{cases}
\frac{d\phi}{dz} = y, \\
\frac{dy}{dz} = \frac{Q(\phi, y)}{f(\phi, y)},
\end{cases} \quad (3)$$

called the second type of singular traveling wave system in [15], where $f(\phi, y)$ and $Q(\phi, y)$ are sufficiently regular functions satisfying the following condition:

$$y \frac{\partial f(\phi, y)}{\partial \phi} + \frac{\partial Q(\phi, y)}{\partial y} \equiv 0, \quad (4)$$

which implies there exists a first integral of Eq. (3). Notice that $\frac{dy}{dz} = \frac{Q(\phi, y)}{f(\phi, y)}$ is not defined on the set of real planar curves $f(\phi, y) = 0$ and when the phase point $(\phi, y)$ passes through every branch of $f(\phi, y) = 0$, the quantity $\frac{dy}{dz}$ changes sign [15]. In this case:

$$f(\phi, y) = \phi^2 \quad \text{and} \quad Q(\phi, y) = k - (pc + \beta)\phi - \phi y^2 - \frac{1}{2}(p + q)\phi^2. \quad (5)$$
The singular curve is $\phi^2 = 0$. We make the coordinate transformation $dz = \phi^2 d\zeta$ for $\phi^2 \neq 0$ to obtain the following regular system associated to (2):

$$\begin{align*}
\frac{d\phi}{d\zeta} &= y\phi^2, \\
\frac{dy}{d\zeta} &= k - (pc + \beta)\phi - \phi y^2 - \frac{1}{2}(p + q)\phi^2.
\end{align*}$$

The systems of equations in (2) and (6) have the same invariant curve solutions, the main difference between Eqs. (2) and (6) is the parametric representation of the orbit: near $\phi = 0$, Eq. (6) uses the fast time variable $\zeta$, while Eq. (2) uses the slow time variable $z$ (see [16,27] for details). Since the first integral of both Eqs. (2) and (6) are the same, thus both of them have the same phase orbits, except on the straight lines $\phi = 0$ and we can study the associated regular system of Eq. (6) in order to get the phase portraits of Eq. (2).

Via standard linear stability analysis we compute the following equilibria of the system (6) when $p + q \neq 0$:

$$P_\pm \equiv \left(\frac{-(pc + \beta) \pm \sqrt{(pc + \beta)^2 + 2k(p + q)}}{p + q}, 0\right).$$

The equilibria $P_\pm$ in (7) exist real when $(pc + \beta)^2 + 2k(p + q) \geq 0$. To test their stability, we compute the jacobian matrix associated to the system (6) evaluated in $P_\pm$:

$$
\begin{pmatrix}
0 & \phi_\pm^2 \\
-(pc + \beta) - (p + q)\phi_\pm & 0
\end{pmatrix},
$$

where $\phi_\pm$ is the first coordinate of the point $P_\pm$. It is straightforward to show that $P_+$, when it exists (i.e. for $k \geq -\frac{(pc + \beta)^2}{2(p + q)}$), is a center and $P_-$, when it exists (i.e. for $k \geq -\frac{(pc + \beta)^2}{2(p + q)}$), is a saddle, see the phase portrait in Fig.1 (the parameters correspond to Fig.14b of [29]).

The oval curve on the right of the singular line $\phi = 0$ in Fig.1 gives rise to a smooth periodic wave solution (compacton) of the original equation. The family of open curves which approaches on the left of the singular straight line $\phi = 0$ gives rise to uncountably infinitely many bounded breaking wave solutions. Moreover, the stable and unstable manifolds of the saddle point $P_-$ gives rise to a one-sided breaking kink wave solution and a one-sided breaking anti-kink wave solution.

When $p + q \neq 0$ and $k = -\frac{(pc + \beta)^2}{2(p + q)}$, the two equilibria $P_+$ and $P_-$ coincide.
Fig. 1. The phase portraits of system (6) when \( p + q \neq 0 \) and \( k \geq -\frac{\beta (pc + \beta)^2}{2(p+q)} \). The parameters are chosen as \( p = 1, q = 2, k = 1, c = 1, \beta = 1 \) and the equilibria are the center \( P_+ \equiv (0.3874, 0) \) and the saddle \( P_- \equiv (-1.7208, 0) \). The straight line \( \phi = 0 \) is drawn in red.

and they are equal to \( P \equiv \left( -\frac{pc + \beta}{p+q}, 0 \right) \). In this case the linear stability analysis fails, as the eigenvalues are equal to zero and the phase portrait is presented in Fig. 2 (for parameters corresponding to Fig. 1) of [29]). The family of open

Fig. 2. The phase portraits of system (6) when \( p + q \neq 0 \) and \( k = -\frac{\beta (pc + \beta)^2}{2(p+q)} \). The parameters are chosen as \( p = 0.5, q = 0.5, c = 1, \beta = 1, k = -1.125 \) and the equilibrium is \( P \equiv (-1.5, 0) \). The straight line \( \phi = 0 \) is drawn in red.

curves approaching the singular line from the right give rise to uncountably infinitely many bounded breaking wave solutions.
When \( p + q = 0 \) the regular system reduces to:

\[
\begin{align*}
\frac{d\phi}{d\zeta} &= y\phi^2, \\
\frac{dy}{d\zeta} &= k - (pc + \beta)\phi - \phi y^2,
\end{align*}
\]

(9)

and, when \( pc + \beta \neq 0 \) the only equilibrium is \( P \equiv \left( \frac{k}{pc+\beta}, 0 \right) \). The characteristic equation corresponding to this equilibrium is straightforwardly obtained as follows:

\[
\lambda^2 + \frac{k^2}{pc + \beta} = 0,
\]

therefore \( P \) is a center when \( pc + \beta > 0 \) and it is a saddle when \( pc + \beta < 0 \), see the corresponding phase portraits in Fig. 3. Notice that in Fig.(3)(a) a limit cycle arises corresponding to the time-dependent first integral. Moreover, the family of open curves which approaches on the left of the singular straight line \( \phi = 0 \) gives rise to uncountably infinitely many bounded breaking wave solutions. The parameters in Fig.(3)(a) to those in Fig. 10 of [29].

In Fig.(3)(b), where the parameters correspond to Fig. 1b) of [29], the stable and unstable manifolds of the saddle point \( P \) gives rise to a one-sided breaking kink wave solution and a one-sided breaking anti-kink wave solution.

![Fig. 3](image)

**Fig. 3.** The phase portraits of system (6) when \( p + q = 0 \). The straight line \( \phi = 0 \) is drawn in red. (a) \( pc + \beta > 0 \). The parameters are chosen as \( p = 0.5, q = -0.5, k = 1, c = 1, \beta = 0.2 \) and the equilibrium is the center \( P \equiv (1.4286, 0) \). (b) \( pc + \beta < 0 \). The parameters are chosen as \( p = 0.5, q = -0.5, k = 1, c = 1, \beta = -0.8 \) and the equilibrium is the saddle \( P \equiv (-3.33, 0) \).

When \( p + q = 0 \) and \( pc + \beta = 0 \) the regular system (6) does not admit any equilibrium and the typical phase portrait is reported in Fig.4. The family
Fig. 4. The phase portraits of system (6) when \( p + q = 0 \) and \( pc + \beta = 0 \). The parameters are chosen as \( p = 0.5, q = -0.5, k = 1, c = 2, \beta = -1 \). The singular straight line \( \phi = 0 \) is drawn in red.

of open curves in Fig. 4 approaching the singular line from the left gives rise to uncountably infinitely many bounded breaking wave solutions. The parameters in Fig.4 correspond to Fig. 1 of [29].

3 Regular pulse and front solutions of the exROE: analytic solutions for homoclinic orbits

In this section, we change gears and consider regular pulse and front solutions of the exROE in (1) by calculating convergent, multi-infinite series solutions for the possible homoclinic orbits. We employ a recently developed approach [7,31], using the method of undetermined coefficients to derive convergent analytic series for homoclinic orbits of Eq. (1).

When \( p + q \neq 0 \) the point \( P_0 \) is a saddle, and the phase portrait of the system (6) shows a homoclinic orbit to this point, see Fig.5 for typical parameters.

We look for a solution of the following form:

\[
\phi(z) = \begin{cases} 
\phi^+(z) & z > 0 \\
x_0 & z = 0 \\
\phi^-(z) & z < 0 
\end{cases} 
\]

(10)

where:

\[
\phi^+(z) = x_0 + \sum_{h=1}^{\infty} a_h e^{h\alpha z}, \quad \phi^-(z) = x_0 + \sum_{h=1}^{\infty} g_h e^{h\gamma z}, 
\]

(11)
Fig. 5. The phase portraits of system (6) when \( p+q \neq 0 \). The parameters are chosen as \( p = 0.5, q = 0.5, k = -1, c = 1, \beta = 1 \). The equilibrium \( P_- \equiv (-2, 0) \) is a saddle and the equilibrium \( P_+ \equiv (-1, 0) \) is a center.

and \( z = x - ct, x_0 \) is the equilibrium point, \( \alpha < 0 \) and \( \gamma > 0 \) are undetermined constants and \( a_h, g_h \), with \( h \geq 1 \), are, at the outset, arbitrary coefficients. Substituting the series (11) for \( \phi^+(z) \) we obtain the following expressions for each term of (1):

\[
\phi^2 = \sum_{h=2}^{\infty} \sum_{j=1}^{h-1} a_{h-j} a_j e^{h\alpha z} + 2x_0 \sum_{h=1}^{\infty} a_h e^{h\alpha z} + x_0^2, \tag{12}
\]

\[
\phi \phi_z^2 = \sum_{h=3}^{\infty} \sum_{j=2}^{h-1} \sum_{l=1}^{j-1} a_l a_{j-l} a_{h-j}(j-l)\alpha^2 e^{kaz} + x_0 \sum_{h=2}^{\infty} \sum_{j=1}^{h-1} (h-j)\alpha^2 a_{h-j} a_j e^{kaz}, \tag{13}
\]

\[
\phi_{zz} \phi^2 = \sum_{h=3}^{\infty} \sum_{j=2}^{h-1} \sum_{l=1}^{j-1} a_l a_{j-l} a_{h-j}(h-j)^2\alpha^2 e^{kaz} + x_0^2 \sum_{h=1}^{\infty} a_h (h\alpha^2) e^{h\alpha z} + 2x_0 \sum_{h=2}^{\infty} \sum_{j=1}^{h-1} (h-j)^2\alpha^2 a_{h-j} a_j e^{kaz}. \tag{14}
\]

Using (12)-(14) into the Eq. (1) we obtain:
\[
\frac{1}{2}(p + q)x_0^2 + (pc + \beta)x_0 - k + \sum_{h=1}^{\infty} (x_0^2(h\alpha)^2 + (p + q)x_0 + pc + \beta) a_h e^{h\alpha z} \\
+ \sum_{h=2}^{\infty} \sum_{j=1}^{h-1} (x_0(h - j))j\alpha^2 + 2x_0(h - j)^2\alpha^2 + \frac{1}{2}(p + q)a_{h-j}a_j e^{h\alpha z} \\
+ \sum_{h=3}^{\infty} \sum_{j=2}^{h-1} \sum_{l=1}^{j-1} ((h - j)^2 + (j - l)l)a_l a_{j-l} a_{h-j} \alpha^2 e^{h\alpha z} = 0. 
\]

As \(x_0\) is an equilibrium of the equation (1), the first three terms in (15) are identical to zero. Comparing the coefficients of \(e^{h\alpha z}\) for each \(h\), one has for \(h = 1\):
\[
(\alpha^2x_0^2 + (p + q)x_0 + pc + \beta)a_1 = 0. 
\]
Assuming \(a_1 \neq 0\) (otherwise \(a_h = 0\) for all \(h > 1\) by induction), results in the two possible values of \(\alpha\):
\[
\alpha_1 = \sqrt{-\frac{(p + q)x_0 + pc + \beta}{x_0^2}}, \quad \alpha_2 = -\sqrt{-\frac{(p + q)x_0 + pc + \beta}{x_0^2}}. 
\]

We are dealing with the case when the equilibrium \(x_0\) is a saddle (see the first section for details), therefore it results that \(\alpha_1 > 0\) and \(\alpha_2 < 0\). In this case, as our series solution (10) needs to converge for \(z > 0\), we pick the negative root \(\alpha = \alpha_2\). For \(h = 2\) we have:
\[
F(2\alpha_2)a_2 = -(3\alpha_2^2x_0 + \frac{1}{2}(p + q)a_1^2, 
\]
where \(F(h\alpha_2) = (h\alpha_2)^2x_0 + (p + q)x_0 + pc + \beta\). For \(h = 3\) we obtain:
\[
a_3 = -\frac{(14\alpha_2^2x_0 + p + q)a_1a_2 + 2\alpha^2a_1^3}{F(3\alpha_2)}. 
\]
And so on, for \(h > 3\) one has:
\[
a_h = \frac{1}{F(h\alpha_2)} \sum_{h=4}^{\infty} \left[ \sum_{j=1}^{h-1} (x_0(h - j))j\alpha^2 + 2x_0(h - j)^2\alpha^2 + \frac{1}{2}(p + q)a_{h-j}a_j \\
+ \sum_{j=2}^{h-1} \sum_{l=1}^{j-1} ((h - j)^2 + (j - l)l)a_l a_{j-l} a_{h-j} \alpha^2 \right]. 
\]

Therefore for all \(h\) the series coefficients \(a_h\) can be iteratively computed in terms of \(a_1\):
\[
a_h = \varphi_h a_1^h, 
\]
9
where $\varphi_h, h > 1$ are functions which can be obtained using Eq. (18)-(20). The first part of the homoclinic orbit corresponding to $z > 0$ has thus been determined in terms of $a_1$:

$$\phi^+(z) = x_0 + a_1 e^{a_2 z} + \sum_{h=2}^{\infty} \varphi_h a_1^h e^{h a_2 z}.$$  \hspace{1cm} (22)

Notice that the Eq. (1) is reversible under the standard reversibility of classical mechanical systems:

$$z \to -z, \quad (\phi, \phi_z, \phi_{zz}) \to (\phi, -\phi_z, \phi_{zz}).$$  \hspace{1cm} (23)

Mathematically, this property would translate to solutions having odd parity in $z$. Therefore the series solution for $z < 0$ can be easily obtained based on the intrinsic symmetry property of the equation, i.e.:

$$\phi^-(z) = x_0 - a_1 e^{a_2 z} - \sum_{h=2}^{\infty} \varphi_h a_1^h e^{h a_2 z}.$$  \hspace{1cm} (24)

We want to construct a solution continuous at $z = 0$, therefore we impose:

$$x_0 + a_1 + \sum_{h=2}^{\infty} \varphi_h a_1^h = 0.$$  \hspace{1cm} (25)

Hence we choose $a_1$ as the nontrivial solutions of the above polynomial equation (25). In practice the Eq. (25) is numerically solved and the corresponding series solutions are not unique.

Let us now choose the system parameter as in Fig.5. Following the above given computation of the series coefficients, we build the homoclinic orbit to the saddle point $P_- \equiv (-2,0)$. Truncating the series solution up to $h = 20$, the corresponding homoclinic orbit solution is unique as the continuity condition (25) admits only the solution $a_1 = 1.5595$ leading to a convergent series coefficients $a_k$, see Fig.6(b), and the continuous series solution for the homoclinic orbit appears as in Fig.6(a), where its traveling nature is also shown.

4 Lagrangian via Jacobi’s Last Multiplier

In this section, we derive a Lagrangian for the traveling wave equation (1) of the exROE equation. While this may be done by simply matching the terms in this equation to those in the Euler-Lagrange equation, we use an alternative approach here using the technique of Jacobi’s Last Multiplier. In Section 5, this Lagrangian will be employed to construct solitary wave solutions of the
Fig. 6. The parameter are chosen as in Fig.5. (a) The series solution \( \phi(z) \) in (10) for the homoclinic orbit to the saddle point \((-2,0)\) plotted as a function of \( x \) for different values of \( t \), showing traveling wave nature of the solution. Here \( a_1 = 1.5595 \) is the only solution of the continuity equation (25) truncated at \( M = 20 \). (b) Plot of \( a_h \) in (20) versus \( h \) shows the series coefficients are converging.

exROE equation, having amplitude and width parameters optimized to satisfy the corresponding Euler-Lagrange equations.

Jacobi [12] first described his method for the “Last Multiplier” (which we shall refer to as the Jacobi Last Multiplier, or JLM for short) in Konigsberg over 1842 – 1843. It essentially yields an extra first integral for dynamical systems by locally reducing an \( n \)-dimensional system to a two-dimensional vector field on the intersection of the \( n - 2 \) level sets formed by the first integrals. After the work of Jacobi, the JLM received a fair amount of attention, including in a classic paper by Sophus Lie [17] placing it within his general framework of infinitesimal transformations. In 1874 Lie [17] showed that one could use point symmetries to determine last multipliers. A clear formulation in terms of solutions or first integrals and symmetries is given by L.Bianchi [33].

Subsequently it was used for computing first integrals of some ordinary differential equations (ODEs). The relation between the Jacobi multiplier denoted by \( M \), and the Lagrangian \( L \) for any second-order ODE was derived by Rao [18], following some investigations in the early twentieth century [34]. After Rao’s work, the JLM does not appear to have been extensively employed in work on dynamical systems till it was recently used by Leach and Nucci to derive Lagrangians for a variety of ODE systems [19,23,24]. Recently more geometric formulation of JLM has been studied in [6].

The study of isochronous behavior, i.e. periodic behavior with a single period, in dynamical systems has also been a subject of great interest over the past decade [1,2]. One important reason for this has been the surprising fact that many, if not most, systems may be converted to nearby isochronous systems by
a process of so-called $\omega$-modification. There are also recent theoretical results [3] proving that, up to a translation or the addition of a constant, planar polynomial systems exhibiting isochronicity are described by either the linear simple harmonic oscillator potential or the isotonic potential.

In this section, we first use the JLM to derive a Lagrangian for the traveling-wave equation of the exROE equation. And then we also investigate possible isochronous behavior in this traveling-wave equation (corresponding to singly-periodic wavetrains of the exROE PDE), we therefore also attempt to map the potential term to either the simple harmonic oscillator (SHO) or the isotonic potential for specific values of the coefficient parameters of the exROE equation.

4.1 Derivation of the Lagrangian via the JLM

Given a $m$-dimensional system of first order ODEs $y'_i = f_i(x, y_i), i = 1, \ldots, m$, the Jacobi last multiplier, denoted by $M(x, y_i)$, is defined as an integrating factor of the system satisfying the following equation:

$$\frac{d(\log M)}{dx} + \sum_{i=1}^{m} \frac{\partial f_i(x, y_i)}{\partial y_i} = 0,$$

Since a second-order ODE $y'' = f(x, y, y')$ is equivalent to a 2-dimensional system of first order ODEs, the corresponding Jacobi multiplier $M(x, y, y')$ satisfies the following equation:

$$\frac{d(\log M)}{dx} + \frac{\partial f(x, y, y')}{\partial y'} = 0,$$

see for details [21,22,24,20,30].

Let us rewrite the Euler-Lagrange equation:

$$\frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = \frac{\partial L}{\partial y},$$

by inserting $y'' = f(x, y, y')$ as follows:

$$\frac{\partial^2 L}{\partial x \partial y'} + y' \frac{\partial^2 L}{\partial y \partial y'} + f(x, y, y') \frac{\partial^2 L}{\partial y^2} = \frac{\partial L}{\partial y}.$$

Assuming $\frac{\partial^2 L}{\partial y^2} \neq 0$ and differentiating equation (29) with respect to $y'$, the
following equation is obtained:

$$\frac{d}{dx} \log \left( \frac{\partial^2 L}{\partial y'^2} \right) + \frac{\partial f}{\partial y'} = 0. \quad (30)$$

Comparing the equation (30) with (27), we find the equation which connects the JLM to the Lagrangian $L$ [18,34,21]:

$$M = \frac{\partial^2 L}{\partial y'^2}. \quad (31)$$

Therefore, the appropriate Lagrangian for the system can be determined starting from the JLM.

4.2 Search for isochronous behavior via the JLM

Isochronous systems, whose motions are periodic with a single period in extended regions of phase-space (often the entire phase-space) have attracted significant interest in recent years, especially following the work of Calogero and his collaborators (see [1,2] and references therein), which revealed the near-ubiquity of such dynamics “close” to numerous classes of dynamical systems. In addition, in [3] it is proved that, up to a possible translation and the addition of a constant, planar polynomial systems exhibiting isochronicity are described by either the linear SHO potential $V(x) = \frac{\omega^2 x^2}{2}$, or the isotonic potential $V(x) = \frac{\omega^2 x^2}{8} + \frac{c^2}{x^2}$. These are rational potential functions, and systems which may be mapped to them exhibit oscillatory solutions with the same period $T = \frac{2\pi}{\omega}$. Irrational potentials, such as some with discontinuous second derivatives, may also be isochronous.

In [8,11] Chouikha and Hill et al. studied conditions under which the so-called Cherkas system [4] with a center at the origin as well as a five-parameter of reversible cubic systems may exhibit isochronicity. However, the study of the isochronicity conditions is non-trivial, and the technique required considerable computational effort. The same problem was re-examined in [5,9] using the JLM to derive the conditions for isochronous solution behavior much more directly and with far less computational effort (see [10] for complete review). Here we shall follow this latter approach to examine (1) for possible isochronous behavior.

Once derived a Lagrangian via the use of the JLM, the next step is to attempt a transformation of variables which might map the Hamiltonian to that of the linear SHO or the isotonic potential. As discussed above, such a mapping would prove isochronous behavior of the original dynamical system [3].
4.3 Lagrangian for the exROE traveling-wave equation

In this subsection, we consider (1) for the traveling waves of the exROE equation. To compute the JLM, we use the equation (27) which, for the exROE, becomes:

$$\frac{d(\log M)}{dz} - \frac{2}{\phi} \phi' = 0. \quad (32)$$

The solution of the equation (32) is given by:

$$M(\phi) = \phi^2. \quad (33)$$

Using the equation (31), we find the appropriate Lagrangian for (1) to be:

$$L(\phi, \phi') = \frac{1}{2} \phi'^2 - (p + q) \frac{\phi^3}{6} - (pc + \beta) \frac{\phi^2}{2} + k\phi, \quad (34)$$

where the potential energy $V(u)$ satisfies the following equation:

$$V'(\phi) = (p + q)\phi^2/2 + (pc + \beta)\phi - k. \quad (35)$$

Applying a Legendre transformation to the Lagrangian $L$ in (34), one can find the corresponding Hamiltonian to be:

$$H = \frac{1}{2} \left( \frac{p}{\sqrt{M(\phi)}} \right)^2 + V(\phi), \quad (36)$$

where the conjugate momentum $p = \frac{\partial L}{\partial \phi'} = M(\phi)\phi'$. 

Next, let us search for isochronous behavior via the use of the JLM. If such behavior were found, it would correspond to period traveling wavetrains in the exROE equation.

Define the canonical variables:

$$P = \frac{p}{\sqrt{M}} \quad \text{and} \quad Q = Q(\phi) \quad (37)$$

to be some function of $\phi$ such that the Poisson bracket $[P, Q] = [p, q]$ is invariant. This implies that $Q'(\phi) = \sqrt{M(\phi)}$. Assuming that there exists a linearizing transformation such that $V(\phi) \to Q(\phi)^2/2$, implies that $V'(\phi) = Q(\phi)Q'(\phi) = \sqrt{M(\phi)}Q(\phi)$, so that:

$$Q(\phi) = \frac{V'(\phi)}{\sqrt{M(\phi)}} = 2\phi. \quad (38)$$
Integrating $Q'(\phi) = \sqrt{M(\phi)}$, we obtain the following equation:

$$Q(\phi) = (p + q)\phi^2 + (pc + \beta) - \frac{k}{\phi}.$$  \hfill (39)

Since $Q$ cannot be determined uniquely, it cannot be a canonical variable. Thus, the potential cannot be directly mapped to a linear harmonic oscillator. Thus, at least within the framework of this method, we do not find any parameter sets for the parameter $c$ for which the exROE traveling-wave equation (1) has isochronous solutions corresponding to singly-periodic traveling wavetrains of the exROE NLPDE.

Hence, we turn next to the construction of solitary waves of the exROE equation using a variational approach.

5 Variational Formulation

5.1 The variational approximation for regular solitons

The procedure for constructing regular solitary waves with exponentially decaying tails is well-known. It is widely employed in many areas of Applied Mathematics and goes by the name of the Rayleigh-Ritz method. In this section, we shall employ it to construct regular solitary waves of (1).

The localized regular solitary wave solutions will be found by assuming a Gaussian trial function:

$$u = A \exp \left( -\frac{z^2}{\rho^2} \right)$$  \hfill (40)

and substituting (40) into the Lagrangian (34).

Note that it is standard to use such Gaussian ansätze for analytic tractability. This is true even for simpler nonlinear PDEs where exact solutions may be known, and have the usual $sech$ or $sech^2$ functional forms. The exponential trial function typically captures these more exact solitary wave forms extremely well in the core or central region of the soliton, with the two often being indistinguishable when plotted together. However, the accuracy is typically somewhat worse in the tails, sometimes with errors of up to a few percent there.

Next, substituting the trial function into the Lagrangian and integrating over all space yields the ‘averaged Lagrangian’ or action (41):
\[
\frac{A\sqrt{\pi}}{72\rho} (9A^3 - 4A^2 \sqrt{3}(p + q)\rho^2 - 18A\sqrt{2}\rho^2 (pc + \beta) + 72k\rho^2). \tag{41}
\]

The next step is to optimize the trial functions by varying the action with respect to the trial function parameters, viz. the core amplitude \(A\), and the core width \(\rho\). This determines the optimal parameters for the trial function or solitary wave solution, but within the particular functional form chosen for the trial function ansatz, in this case a Gaussian. The resulting variational Euler-Lagrange equations, by varying \(A\) and \(\rho\) respectively, are the system of algebraic equations:

\[
3A^3 - \sqrt{3}(p + q)\rho^2 A^2 - 3\sqrt{2}(pc + \beta)\rho^2 A + 6k\rho^2) = 0, \tag{42}
\]
\[
9A^3 + 4\sqrt{3}(p + q)\rho^2 A^2 + 18\sqrt{2}(pc + \beta)\rho^2 A - 72k\rho^2 = 0. \tag{43}
\]

Given their relative simplicity, and assuming \(a_1 = 1/2, a_3 = 1\), a nontrivial solution to equations (42)-(43) is:

\[
A = \frac{(-9\sqrt{6}(pc + \beta) + \sqrt{840\sqrt{3}(p + q)k + 486(pc + \beta)^2})}{14(p + q)}, \tag{44}
\]
\[
\rho^2 = \frac{3A^3}{(3A(pc + \beta)\sqrt{2} + A^2(p + q)\sqrt{3} - 6k)}. \tag{45}
\]

The optimized variational soliton for the regular solitary waves of the traveling-wave equation (1) is given by the trial function (40) with the above \(A\) and \(\rho\). Figure 7 shows the resulting regular solitary wave solution for various values of the parameters.

Figure 8 shows a direct analysis of the accuracy of the variational regular solitary waves obtained above. In this instance, we are able to do a direct accuracy analysis since our variational solution for the regular solitary waves given by (40), (44) and (45) is, unlike for most variational solutions, an analytical one. Inserting this variational solution (40) (with (44) and (45)) into the traveling-wave ODE (1), the deviation of the left-hand side of (1) from zero gives a direct measure of the goodness of the variational solution.

Figure 8 shows this left-hand side for \(p = 1, q = 1, \beta = 0.5\) and \(k = 2\). For positive values of the wave-speed \(c\), the error is small. However, the error grows rapidly for \(c\) negative.
Fig. 7. (a) The regular soliton for $p = 1, q = 1, \beta = 0.5$ and $k = 2$, for different values of $c$. (b) The regular soliton for $p = 1, q = 1, \beta = 0.5$ and $c = 2$, for different values of $k$. (c) The regular soliton for $p = 1, \beta = 0.5, c = 2$ and $k = 2$, for different values of $q$. (d) The regular soliton for $p = 1, q = 1, c = 2$ and $k = 2$, plotted for different values of $\beta$.

5.2 The variational approximation for embedded solitons

In the recent and novel variational approach to embedded solitary waves, the tail of a delocalized soliton is modeled by:

$$u_{\text{tail}} = \alpha \cos(\kappa(c)z).$$

(46)

Our embedded solitary wave will be embedded in a sea of such delocalized solitons. The cosine functional form ensures an even solution, and the arbitrary function $\kappa(c)$ will, as shown below, help to ensure the integrability of the action. Our ansatz for the embedded soliton uses a second order exponential core model plus the above tail model [13,14]:
Plugging this ansatz into the Lagrangian (34) and reducing the trigonometric powers to double and triple angles yields an equation with trigonometric functions of the double and triple angles, as well as terms linear in \( z \). The former would make spatial integration or averaging of the Lagrangian divergent. However, it is possibly to rigorously establish, following a procedure analogous to proofs of Whitham’s averaged Lagrangian technique [32], that such terms may be averaged out, so we shall set them to zero \textit{a priori}.

The terms linear in \( z \) would also cause the Lagrangian to be non-integrable. To suppress these, we therefore set:

\[
\kappa(c) = \pm \frac{2\sqrt{pc + \beta}}{\alpha},
\]

which makes linear terms zero. Note that this step, and the preceding step of averaging out trigonometric functions of the higher angles are recent ones for the variational approximation of embedded solitary waves. They are not part of the traditional Rayleigh-Ritz method used for the construction of regular solitary waves.

Next, the rest of the equation can be integrated to give the action:
As for the regular solitary waves, the action is now varied with respect to the core amplitude $A$, the core width $\rho$, and the small amplitude $\alpha$ of the oscillating tail. For strictly embedded solitary waves, which occur on isolated curves in the parameter space where a continuum or “sea” of delocalized solitary waves exist, the amplitude of the tail is strictly zero. Once again, this is an extra feature not encountered in the standard variational procedure. Hence, we also need to set $\alpha = 0$ in these three variational equations to recover such embedded solitary waves. Implementing this, we have:

\[
\begin{align*}
\frac{1}{8\rho} & \left( \left( -2(\beta - \frac{1}{2}\kappa^2 A^2 + pc) \rho^2 - \frac{\alpha^2}{2} \right) \sqrt{2}A - \frac{4\sqrt{3}}{9} A^2 \rho^2 (p + q) + \right. \\
& \left. \quad (-2(p + q)\alpha^2 + 8k)\rho^2 + A^3 \right) e^{\frac{95}{16}\kappa^2 \rho^2} + \frac{16}{27} \alpha \left( -\frac{27}{2} \left( \beta - \frac{\kappa^2 A^2}{2} + pc \right) \rho^2 e^{\frac{89}{16}\kappa^2 \rho^2} \\
& \quad - \frac{27}{8} \rho^2 (p + q) e^{\frac{\kappa^2 A^2}{8}} (\alpha e^{\frac{71}{16}} + \sqrt{2} A e^{23}) + \frac{27\sqrt{2}}{8} \alpha \left( \kappa^2 A^2 + \frac{1}{2} \right) \rho^2 e^{\frac{89}{16}\kappa^2 \rho^2} \\
& \quad + \sqrt{3} A^2 (\kappa^2 \rho^2 + 3) e^{\frac{31}{16}\kappa^2 \rho^2} + \frac{27}{4} \alpha^2 \rho^2 \kappa^2 e^{\frac{41}{16}\kappa^2 \rho^2} \right) e^{-\frac{95}{16}\kappa^2 \rho^2 \sqrt{\pi}A}.
\end{align*}
\] (49)

Subtracting the two equations (50), (51), one may obtain an expression for $A$ in terms of $\rho$. Solving the two equations obtained by substituting this expression for $A$ into the equation (50) and the equation (52) yields the solutions $c = 0$, $\rho^2 = constant$. Thus, no non-trivial embedded soliton solutions result for the exROE equation.

As for the regular solitary waves, the action is now varied with respect to the core amplitude $A$, the core width $\rho$, and the small amplitude $\alpha$ of the oscillating tail. For strictly embedded solitary waves, which occur on isolated curves in the parameter space where a continuum or “sea” of delocalized solitary waves exist, the amplitude of the tail is strictly zero. Once again, this is an extra feature not encountered in the standard variational procedure. Hence, we also need to set $\alpha = 0$ in these three variational equations to recover such embedded solitary waves. Implementing this, we have:

\[
\begin{align*}
\rho^2 \left( 6k - 3\sqrt{2}(pc + \beta)A - \sqrt{3}(p + q)A^2 \right) + 3A^3 &= 0 \quad (50) \\
\rho^2 \left( -72k + 18\sqrt{2}(pc + \beta)A + 4\sqrt{3}(p + q)A^2 \right) + 9A^3 &= 0 \quad (51) \\
-8\sqrt{3} A^2 (\rho^2 \kappa^2 + 3) \exp \left( \frac{31\rho^2 \kappa^2}{8} \right) + 108(pc + \beta)\rho^2 \exp \left( \frac{89\rho^2 \kappa^2}{24} \right) \\
& \quad + 27\sqrt{2} A \rho^2 (p + q) \exp \left( \frac{23\rho^2 \kappa^2}{6} \right) = 0 \quad (52)
\end{align*}
\]
6 Conclusions

Three recent analytical approaches have been applied in this paper to treat the possible classes of traveling wave solutions of the so-called extended-reduced Ostrovsky (exROE) equations.

A recent, novel application of phase-plane analysis is first employed to show the existence of breaking kink wave and smooth periodic (compacton) solutions in certain parameter regimes.

Smooth traveling waves are next considered using a recent technique to derive convergent multi-infinite series solutions for the homoclinic orbits of the traveling-wave equations for the exROE equation, as well as for its generalized version with arbitrary coefficients. These correspond to pulse or solitary wave solutions respectively of the original PDE. Unlike the majority of unaccelerated convergent series, high accuracy is attained with relatively few terms. We also show the traveling wave nature of these pulse and front solutions.

Finally, variational methods are employed to treat families of both regular and embedded solitary wave solutions for the exROE PDE. The technique for obtaining the embedded solitons incorporates several recent generalizations of the usual variational technique and is thus topical in itself. One unusual feature of the solitary waves derived here is that we are able to obtain them in analytical form (within the assumed ansatz for the trial functions). Thus, a direct error analysis is performed, showing the accuracy of the resulting solitary waves.

Given the importance of wave solutions in dynamics and information propagation, and the fact that quite little is known about solutions of the family of generalized exROE equations considered here, the results obtained are both new and topical.

References


