The chaotic Dadras-Momeni system: control and hyperchaotification

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Abstract

In this paper a novel three-dimensional autonomous chaotic system, the so called Dadras-Momeni system, is considered and two different control techniques are employed to realize chaos control and chaos synchronization. Firstly, the optimal control of the chaotic system is discussed and an open loop feedback controller is proposed to stabilize the system states to one of the system equilibria, minimizing the cost function by virtue of the Pontryagin’s minimum principle. Then, an adaptive control law and an update rule for uncertain parameters, based on Lyapunov stability theory, are designed both to drive the system trajectories to an equilibrium or to realize a complete synchronization of the system in periodic and chaotic regimes. Numerical simulations are included to show the effectiveness of the designed controllers in realizing the stabilization and the synchronization of the chaotic system. Finally, a new hyperchaotic system is introduced by adding a controller to the second equation of the three-dimensional autonomous Dadras-Momeni chaotic system and the corresponding hyperchaotic attractor is shown via computer simulations.

Keywords: Optimal control; Synchronization; Lyapunov function; Pontryagin Minimum Principle; Multi-scroll chaotic attractor; Hyperchaotic system.

1 Introduction

In the last decades, a great effort has been devoted towards several aspects of nonlinear dynamical systems, in particular to chaotic dynamics. Being a very relevant nonlinear phenomenon, chaos has been intensively investigated in diverse context including mechanical, electrical or chemical processes. Recently, a new three-dimensional autonomous chaotic system has been introduced by Dadras and Momeni in [7]. In each state equation of this system a multiplier term is added introducing the nonlinearity necessary for folding trajectories. This system admits five real equilibria and
has a very rich (and still not completely explored) nonlinear dynamics, including chaos and period doubling bifurcations. Moreover, it can generate two, three and four-scroll chaotic attractors varying one single system parameter. In order to examine the chaotic regimes of the Dadras-Momeni system, we re-interpret the chaotic system as ensembles of competing nonlinear oscillators and we employ the technique of competitive modes analysis as a diagnostic tool to identify possible chaotic regimes [6, 28, 31, 32, 33, 34, 37].

The control of chaos has became an important challenge in chaos theory, including stabilization of chaotic systems to steady states or regular behaviour. In particular, some useful methods have been developed to realize chaos suppression such as linear and nonlinear feedback [15, 18, 39], adaptive control [4, 16, 41], optimal control [8, 9, 11, 12, 20, 29], time delay control [40], intermittent and passive control [17], to name a few. In this paper, we firstly propose a strategy for optimal control of the chaotic Dadras-Momeni system and, applying the Pontryagin Minimum Principle, we design an optimal feedback controller which drives the system states to an equilibrium. Furthermore, an adaptive control scheme is employed to stabilize the chaotic system trajectories to a point. The obtained control in both cases is global, easy to implement and robust and all the performed numerical simulations demonstrate the effectiveness of the proposed control strategies.

The synchronization of two chaotic systems consists into designing a controller or forcing in such a way that the motion of each system can be adjusted to a common dynamics. The intrinsic nature of the chaotic systems does not obey to this idea of synchronization due to the sensitivity to initial conditions; in fact the trajectories of two identical chaotic systems evolve to completely different dynamical behavior when starting from different initial conditions (however they are close). However, the interest in synchronization schemes of chaotic systems is also rapidly increasing [11, 14, 27, 34] due to important application in secure communications [2, 38, 3] and cryptography [35]. Here we design a nonlinear controller according to Lyapunov’s direct method to guarantee a complete and global synchronization between two identical Dadras-Momeni systems in their chaotic regime. Two numerical examples are presented in order to illustrate this synchronization.

Finally, as it is believed that chaotic systems with higher dimensional attractors may have much wider application (the adoption of higher dimensional chaotic systems has been proposed for secure communication and the presence of more than one positive Lyapunov exponent clearly improves security of the communication scheme by generating more complex dynamics), hyperchaos has attracted increasing attention from several scientific and engineering communities [5, 10, 13, 19, 26, 30]. Here we generate a new hyperchaotic system by adding a controller to the second equation of the three-dimensional chaotic Dadras-Momeni system. The computation of the full Lyapunov spectrum, as one system parameter varies, shows that the obtained system support hyperchaotic behavior over a large range.

This paper is organized as follows. In Section 2, we briefly analyze the main properties of the chaotic Dadras-Momeni system and we also apply the competitive modes analysis as a diagnostic tool to determine parameter regimes in which chaos might occur. In Section 3, we propose two different feedback controllers to stabilize the chaotic Dadras-Momeni system to an equilibrium; in particular an optimal control is solved applying the Pontryagin Minimum Principle and an adaptive scheme is employed to globally drive the chaotic system trajectories to its steady states. In Section 4, the complete synchronization of two Dadras-Momeni system in their chaotic regime is realized via adaptive control. In Section 5 the new hyperchaotic system is introduced and analyzed.
2 The Dadras-Momeni system: linear stability analysis and competitiveness of modes

In this section, we briefly introduce the chaotic Dadras-Momeni system and its main properties (for a detailed description of the system see [7]). Moreover, as the existence of positive Lyapunov exponent is not always indication of chaos (see [22] and [21]), we use the competitive modes conjecture as it provides a necessary condition to diagnostic chaos occurrence.

2.1 Linear analysis

Dadras and Momeni reported a new chaotic system in [7], which is now called the Dadras-Momeni system. The system is described by the following nonlinear differential equations:

\[
\begin{align*}
\dot{x} &= y - ax + byz, \\
\dot{y} &= cy - xz + z, \\
\dot{z} &= dxy - hz,
\end{align*}
\] (2.1)

where \(x, y\) and \(z\) are system state variables and \(a, b, c, d\) and \(h\) are positive constant parameters. This new system is found to be chaotic in a wide range of parameters and it supports many interesting complex dynamical behaviors. For example, the system (2.1) generates a new two-wing chaotic attractor, shown in Fig.1, for the following parameter set:

\(a = 3, \quad b = 2.7, \quad c = 4.7, \quad d = 2\) and \(h = 9\). (2.2)

There exist five possible equilibrium values for the Dadras-Momeni system (2.1):

\[
\begin{align*}
E_1 &\equiv (0, 0, 0), \\
E_2 &\equiv \left(\frac{d + \sqrt{\Delta_1}}{2d}, \frac{b}{b} \left(\frac{-1 + \sqrt{\Delta_2}}{d + \sqrt{\Delta_1}}, \frac{-1 + \sqrt{\Delta_3}}{2b}\right), \\
E_3 &\equiv \left(\frac{d + \sqrt{\Delta_1}}{2d}, \frac{b}{b} \left(\frac{-1 - \sqrt{\Delta_2}}{d + \sqrt{\Delta_1}}, \frac{-1 - \sqrt{\Delta_3}}{2b}\right), \\
E_4 &\equiv \left(\frac{d - \sqrt{\Delta_1}}{2d}, \frac{b}{b} \left(\frac{-1 + \sqrt{\Delta_2}}{d - \sqrt{\Delta_1}}, \frac{-1 + \sqrt{\Delta_3}}{2b}\right), \\
E_5 &\equiv \left(\frac{d - \sqrt{\Delta_1}}{2d}, \frac{b}{b} \left(\frac{-1 - \sqrt{\Delta_2}}{d - \sqrt{\Delta_1}}, \frac{-1 - \sqrt{\Delta_3}}{2b}\right),
\end{align*}
\] (2.3)

where:

\[
\Delta_1 = d^2 + 4chd, \quad \Delta_2 = \frac{2ab}{h}(d + 2ch + \sqrt{\Delta_1}), \quad \Delta_3 = \frac{2ab}{h}(d + 2ch - \sqrt{\Delta_1}).
\]

As they will be used in the following of the paper, here we report the values of the equilibria in (2.3) when the parameters are chosen as in (2.2):

\[
\begin{align*}
E_1 &\equiv (0, 0, 0), \\
E_2 &\equiv (5.1260, 2.0794, 2.3687), \\
E_3 &\equiv (5.1260, -2.4045, -2.7390), \\
E_4 &\equiv (-4.1260, -2.5834, 1.8734), \\
E_5 &\equiv (-4.1260, 2.9873, -2.2438).
\end{align*}
\] (2.4)
Via a straightforward calculation it is easy to show that the equilibria in (2.4) are all unstable. Moreover, as long as $\nabla V = c - h - a < 0$, and this is the case when the parameters are as in (2.2), the system (2.1) is dissipative and the trajectories converge to a set of measure zero at an exponential rate, i.e. $x(t), y(t)$ and $z(t)$ are globally bounded and an attractor exists.

In [7] some other chaotic regimes have been investigated, showing that the system (2.1) generates two, three, and four-scroll chaotic attractors by varying only the parameter $c$. However, the switching mechanism between the multi-scroll attractors is unknown. Moreover, due to its extremely complicated dynamics, the topological structure of the new system should be investigated completely and also the localization of hidden attractors should be addressed [23, 24, 25].

2.2 Competitiveness of modes

Competitive modes have been introduced in the literature as a sort of chaos diagnostic. More precisely, they are a conjectured necessary condition for the occurrence of chaos. In the present section, we show that for the parameter regime as in (2.2) at least two modes become competitive.

First, we give a bit of an overview of the method. Let us consider the following general nonlinear autonomous system:

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n).$$

(2.5)

Differentiating (2.5) once gives:

$$\ddot{x}_i = \sum_{j=1}^{n} f_j \frac{\partial f_i}{\partial x_j} = -x_i G_i(x_1, x_2, \ldots, x_i, \ldots, x_n)$$

$$+ H_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

(2.6)

We will consider the competitive modes to be solutions $x_i$ of the model system given by (2.5), and, in analogy with systems of coupled parametric oscillators, we shall term the $G_i$ in (2.6) as the squared frequency corresponding to the oscillations of the mode $x_i$. Like these squared frequencies, the $H_i$ (the forcing terms) can also depend on the other mode amplitudes (but not on the $i$th).

In order to apply the theory of competitive modes to determine parameter regimes for which the Dadras-Momeni system may exhibit chaotic behavior, recall that the following is posed in [37] (p. 95):

**Competitive Modes Conjecture**: The conditions for dynamical systems to be chaotic are given by:

(i) there exist at least two modes, labelled $G_i$’s in the system;

(ii) at least two $G$’s are competitive or nearly competitive, that is, for some $i$ and $j$, $G_i \approx G_j > 0$ at some $t$;

(iii) at least one of the $G$’s is a function of evolution variables such as $t$;

(iv) at least one of the $H$’s is a function of system variables.
Let us compute the modes along the lines of system (2.1). Taking time derivatives of (2.1) yields the following system of oscillator equations:

\[\ddot{x} + G_1 x = \ddot{x} + (z + bz^2) x = (c - a)y + (c - a - h)byz + z + bz^2 = H_1,\]

\[\ddot{y} + G_2 y = \ddot{y} + (z + bz^2) + dx(x - 1) - c^2 y = (a + h - c)yz + (c - h)z = H_2,\]

\[\ddot{z} + G_3 z = \ddot{z} + (dx(x - 1) - h^2 - by^2) z = (c - a - h)dx y + dy^2 = H_3.\]

Comparing (2.7) with (2.6), we identify the nonlinear mode frequencies as:

\[G_1 = z + bz^2 - a^2 - bdy^2,\]

\[G_2 = z + bz^2 + dx(x - 1) - c^2,\]

\[G_3 = dx(x - 1) - h^2 - by^2,\]

and clearly, the \(H_k\)'s are not identically zero. In order to satisfy the above given Competitive Modes Conjecture and therefore identify possible regimes leading to chaos, two modes in (2.8) should be competitive at a point, i.e. one of the following conditions on the mode frequencies should hold at some \(t = t^*\):

1) \(G_1 = G_2 \Leftrightarrow \alpha_1(\alpha_1 - 1) + b\alpha_2^2 = (c^2 - a^2)/d;\)

2) \(G_1 = G_3 \Leftrightarrow \alpha_3(1 + b\alpha_3) + h^2 = d\alpha_1(\alpha_1 - 1) + a^2;\)

3) \(G_2 = G_3 \Leftrightarrow \alpha_3(1 + b\alpha_3) + bda_2^2 = c^2 - h^2;\)

where \((\alpha_1, \alpha_2, \alpha_3)^T \equiv (x(t^*), y(t^*), z(t^*))^T.\)

When any of these three conditions are met, the respective modes are competitive and thus chaos is possible. For the parameter set (2.2) with initial conditions \((x(0), y(0), z(0))^T = (5, 0, -4)^T\), we see that two modes become competitive on the domain, with the two mode frequencies becoming equal infinitely often; see Fig. 2. The situation observed in Fig. 2 corresponds to case 2), since both \(G_1\) and \(G_3\) are equal at various points throughout the domain. We have also considered other parameter sets leading to chaotic behaviour as identified in [7], which are not reported here, all confirming the effectiveness of the competitive modes conjecture.

As was pointed out in [32], often two modes are competitive in the presence of chaos, while three modes being competitive can signify hyperchaos in nonlinear systems of dimension greater than three. So, for the present system, we do not expect all three modes to be competitive, and in the cases considered this holds true.

3 Feedback control of the chaotic Dadras-Momeni system

In this section we design two different feedback controllers using first an optimal control and then an adaptive control technique to globally regulates the trajectories of the chaotic system (2.1) to one of its equilibrium points.

Consider the following system:
\[\begin{aligned}
\dot{x} &= y - ax + byz + u_1, \\
\dot{y} &= cy - xz + z + u_2, \\
\dot{z} &= dxy - hz + u_3,
\end{aligned}\]  

(3.1)

where the control input variables \(u_1, u_2\) and \(u_3\) has been added respectively to each equation of the system (2.1). The system (3.1) is supplemented with the initial conditions \((x(0), y(0), z(0)) \equiv (x_0, y_0, z_0)\).

3.1 Optimal control

The goal of this section is to design a state feedback law \(u = (u_1, u_2, u_3)^T\) stabilizing in a given time \(t_f\) the chaotic system (3.1) to one of its equilibrium points \(E_i\) in (2.3) and meanwhile minimizing the following cost function:

\[J = \frac{1}{2} \int_0^{t_f} \left( \sum_{i=1}^{3} (\alpha_i(\phi_i - \bar{\phi}_i)^2 + \beta_i u_i^2) \right) dt,\]  

(3.2)

where \(\alpha_i, \beta_i, (i = 1, 2, 3)\) are positive constants, \(\phi_1 = x, \phi_2 = y, \phi_3 = z\), are the running cost and \(\bar{\phi}_1 = \bar{x}, \bar{\phi}_2 = \bar{y}, \bar{\phi}_3 = \bar{z}\) are the final cost, with \((\bar{x}, \bar{y}, \bar{z})\) denoting the coordinates of one the equilibria \(E_i\) in (2.3). Notice that the cost function (3.2) is a positive definite function of the variables \(\phi_i\), and \(u_i, i = 1, \ldots, 3\), and it is zero just at the point \(\phi_i = \bar{\phi}_i\). In order to solve the optimal control problem we will derive the fundamental nonlinear Two-Point Boundary Value Problem arising in the Pontryagin Minimum Principle (PMP). Consider the Hamiltonian function defined by:

\[H = -\frac{1}{2} \left[ \alpha_1(x - \bar{x})^2 + \alpha_2(y - \bar{y})^2 + \alpha_3(z - \bar{z})^2 + \beta_1 u_1^2 + \beta_2 u_2^2 + \beta_3 u_3^2 \right] + \lambda_1[y - ax + byz + u_1] + \lambda_2[cy - xz + z + u_2] + \lambda_3[dxy - hz + u_3],\]  

(3.3)

where \(\lambda_i, (i = 1, 2, 3)\) are the co-state variables and they are the solutions of the following backward Hamiltonian equations:

\[\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x}, \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial y}, \\
\dot{\lambda}_3 &= -\frac{\partial H}{\partial z}.
\end{aligned}\]  

(3.4)

Substituting (3.3) into (3.4), the co-state equations are explicitly given as follows:

\[\begin{aligned}
\dot{\lambda}_1 &= \alpha_1(x - \bar{x}) + a\lambda_1 + \lambda_2 z - d\lambda_3 y, \\
\dot{\lambda}_2 &= \alpha_2(y - \bar{y}) - \lambda_1 - b\lambda_1 z - c\lambda_2 - d\lambda_3 x, \\
\dot{\lambda}_3 &= \alpha_3(z - \bar{z}) - b\lambda_1 y + \lambda_2 x - \lambda_2 + h\lambda_3.
\end{aligned}\]  

(3.5)
According to the PMP, the optimal control functions that have to be used are determined from the conditions

\[ \frac{\partial H}{\partial u_i} = 0, \quad (i = 1, 2, 3). \]

Hence, we get:

\[ u_i^* = \frac{\lambda_i}{\beta_i}, \quad (i = 1, 2, 3). \tag{3.6} \]

Substituting (3.6) into (3.1) the nonlinear controlled state equations result:

\[
\begin{aligned}
\dot{x} &= y - ax + byz + \frac{\lambda_1}{\beta_1}, \\
\dot{y} &= cy - xz + z + \frac{\lambda_2}{\beta_2}, \\
\dot{z} &= dxy -hz + \frac{\lambda_3}{\beta_3}.
\end{aligned}
\tag{3.7}
\]

supplemented with the following boundary conditions:

\[
\begin{aligned}
x(0) &= x_0, & x(t_f) &= \bar{x}, \\
y(0) &= y_0, & y(t_f) &= \bar{y}, \\
z(0) &= z_0, & z(t_f) &= \bar{z}, \\
\lambda_i(t_f) &= 0, & i &= 1, 2, 3.
\end{aligned}
\tag{3.8}
\]

Solving the nonlinear systems (3.5) and (3.7) with the boundary conditions of (3.8), we obtain the optimal control law and the corresponding optimal state trajectory.

To check the effectiveness of the above given theoretical analysis, we solve the systems (3.7) and (3.8) with initial conditions \((x_0, y_0, z_0) \equiv (5, 0, -4)\) and the system parameters selected as in (2.2) in such a way that the system (3.1) exhibits a chaotic behavior if no control is applied. The initial values of co-states corresponding to each \(E_i (i = 1, 2, \cdots, 5)\) are taken in Table 1. The positive constants in the cost function (3.2) are chosen as \(\alpha_1 = 0.001, \, \alpha_2 = 0.001, \, \alpha_3 = 0.001, \, \beta_1 = 0.0000001, \, \beta_2 = 0.0000001, \, \beta_3 = 0.0000001\). The systems (3.7) and (3.8) are solved using the MATLAB’s bvp4c in-built solver. The resulting dynamical behavior of the controlled system (3.1) is shown in Figs. 3-7.

### 3.2 Adaptive control

In this section, we design an adaptive control law, and an update rule for uncertain parameters based on Lyapunov stability theory, to drive the trajectories of the system (3.1) to one of its equilibria (2.3).

**Theorem 1** The system (3.1) with unknown system parameters is globally and exponentially stabilized to the point \((\bar{x}, \bar{y}, \bar{z})\) by the following adaptive control law:

\[
\begin{aligned}
u_1 &= -y + a_1x - b_1yz - k_1(x - \bar{x}), \\
u_2 &= -c_1y + xz - z - k_2(y - \bar{y}), \\
u_3 &= -d_1xy + h_1z - k_3(z - \bar{z}),
\end{aligned}
\tag{3.9}
\]
and the following parameter estimation update laws:

\[
\begin{align*}
\dot{a}_1 &= -x(x - \overline{a}) + k_4(a - a_1), \\
\dot{b}_1 &= yz(x - \overline{a}) + k_5(b - b_1), \\
\dot{c}_1 &= y(y - \overline{y}) + k_6(c - c_1), \\
\dot{d}_1 &= xy(z - \overline{a}) + k_7(d - d_1), \\
\dot{h}_1 &= -z(z - \overline{z}) + k_8(h - h_1),
\end{align*}
\]

(3.10)

where \(a_1, b_1, c_1, d_1, h_1\) are estimate values of the uncertain parameters \(a, b, c, d, h\), and \(k_i (i = 1, \cdots, 8)\) are positive constants (called control gains).

**Proof.**

Let us define the positive definite Lyapunov function candidate:

\[
V(x, y, z, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{h}) = \frac{1}{2}((x - \overline{a})^2 + (y - \overline{y})^2 + (z - \overline{z})^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{h}^2)
\]

(3.11)

where \(\tilde{a} = a - a_1, \tilde{b} = b - b_1, \tilde{c} = c - c_1, \tilde{d} = d - d_1\) and \(\tilde{h} = h - h_1\).

Substituting the control inputs (3.9) into (3.11), we get the following closed-loop system:

\[
\begin{align*}
\dot{x} &= -(a - a_1)x + (b - b_1)yz - k_1(x - \overline{a}), \\
\dot{y} &= (c - c_1)y - k_2(y - \overline{y}), \\
\dot{z} &= (d - d_1)xy - (h - h_1)z - k_3(z - \overline{z}),
\end{align*}
\]

(3.12)

Calculating the time derivative of the Lyapunov function (3.11) along the trajectories of the controlled system (3.12) we obtain:

\[
\begin{align*}
\dot{V} &= (x - \overline{a})\dot{x} + (y - \overline{y})\dot{y} + (z - \overline{z})\dot{z} + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + \tilde{h}\dot{\tilde{h}} \\
&= -k_1(x - \overline{a})^2 - k_2(y - \overline{y})^2 - k_3(z - \overline{z})^2 + \tilde{a}\dot{a}_1 + (b - b_1)[(x - \overline{a})y - \dot{b}_1] + (c - c_1)[y(y - \overline{y}) - \dot{c}_1] + (d - d_1)[(z - \overline{z})x - \dot{d}_1] \\
&\quad + (h - h_1)[(h - h_1)z - \dot{h}_1].
\end{align*}
\]

(3.13)

Substituting (3.10) into (3.13), the time derivative of the Lyapunov function reads:

\[
\begin{align*}
\dot{V} &= -k_1(x - \overline{a})^2 - k_2(y - \overline{y})^2 - k_3(z - \overline{z})^2 - k_4(a - \overline{a})^2 - k_5(b - \overline{b})^2 - k_6(c - \overline{c})^2 \\
&\quad - k_7(d - \overline{d})^2 - k_8(h - \overline{h})^2 < 0.
\end{align*}
\]

(3.14)

Therefore \(\dot{V}\) is negative definite in the neighborhood of the zero solution of the system (3.12). According to the Lyapunov stability theory, the equilibrium solution \(E(\overline{x}, \overline{y}, \overline{z})\) of the controlled system (3.12) is asymptotically stable, namely the controlled system (3.12) asymptotically converges to the equilibrium \(E(\overline{x}, \overline{y}, \overline{z})\) with the adaptive control laws (3.9) and the parameter estimation update laws (3.10). This completes the proof. \(\square\)

To corroborate the effectiveness of the proposed adaptive control (3.9), we employ the following numerical experiment: the parameters are fixed as in (2.2), therefore the system (3.1) exhibits...
where whose evolution is guided by the controllers and the second one is the slave or response system:

\[ \lim_{t \to \infty} \|e(t)\| = 0, \quad e = [e_1 \; e_2 \; e_3]^T. \]

**Theorem 2** For any initial conditions, the two systems (4.1) and (4.2) are globally asymptotically synchronized by the following control law:

\[
\begin{align*}
  u_1 &= -(k_1 - a_1)e_1 - (1 - b_1z_m - z_s)e_2 \\
  u_2 &= -(k_2 + c_1)e_2 - (1 + d_1 - x_m)e_3 \\
  u_3 &= -(k_3 - h_1)e_3 - (b_1y_s + d_1ym)e_1
\end{align*}
\]

where \( k_i \) are the positive scalar control gains and by the parameter update rules:
where \( e_a = a_1 - a, e_b = b_1 - b, e_c = c_1 - c, e_d = d_1 - d, e_h = k_1 - k \).

**Proof.** Let us choose the following Lyapunov function:

\[
V = \frac{1}{2} \left( e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 \right),
\]

which is positive definite. Calculating the time derivative of the Lyapunov function (4.6) along the trajectories of the error system (4.3) one obtains:

\[
\dot{V} = \begin{pmatrix} e_{a1} = x_m e_1 \\ e_{b1} = -y_m z_m e_1 \\ e_{c1} = -y_m e_2 \\ e_{d1} = -x_m y_m e_3 \\ e_{h1} = z_m e_3 \end{pmatrix}
\]

(4.5)

Substituting equations (4.4) and (4.5) into the expression (4.7) for \( \dot{V} \) and noting that the following equalities hold:

\[
\begin{align*}
-x_s z_s + x_m z_m &= -z_s e_1 - x_m e_3, \\
x_s y_s - x_m y_m &= x_s e_2 + y_m e_1, \\
y_m z_m - y_s z_s &= y_s e_3 - z_m e_2
\end{align*}
\]

(4.7)

one gets:

\[
\dot{V} = -\sum_{i=1}^{3} k_i e_i^2 = -e^T Pe,
\]

where \( P = \text{diag}\{k_1, k_2, k_3\} \).

(4.8)

Since \( \dot{V} \) is negative semidefinite \( e_1, e_2, e_3, e_a, e_b, e_c, e_d, e_h \in \mathcal{L}_\infty \) and from the error system (4.3) it follows that \( \dot{e}_1, \dot{e}_2, \dot{e}_3 \in \mathcal{L}_\infty \). Given \( \lambda_{\min}(P) \) the minimum eigenvalue of the matrix \( P \), one gets:

\[
\int_{0}^{t} \lambda_{\min}(P) ||e||^2 dt \leq \int_{0}^{t} e^T Pedt \leq \int_{0}^{t} -\dot{V} dt = V(0) - V(t) \leq V(0).
\]

(4.9)

Therefore \( e_1, e_2, e_3 \in \mathcal{L}_2 \) and the hypotheses of the Barbalat’s lemma are satisfied.

Thus \( \lim_{t \to \infty} ||e(t)|| = 0 \) and the proof is completed. \( \square \)

To test the effectiveness of the proposed adaptive synchronization scheme, we show two numerical examples in which the Dadras-Momeni system has been chosen in its double-periodic and chaotic regime. By choosing the parameters \( a = 3, b = 2.7, c = 5.55, d = 2, h = 9 \) the master system support a period-doubling bifurcation shown in Fig (13)(b). Let us assume that the starting points for the master and the slave systems are respectively \( (x_m(0), y_m(0), z_m(0)) = (2.5, 0, -1.5) \) and
(x_s(0), y_s(0), z_s(0)) = (-2.5, 0, 1.5). Given the initial conditions for the uncertain parameters of the response system a_1(0) = 3, b_1(0) = 5, c_1(0) = 2, d_1(0) = -30, h_1(0) = -5, the adaptive control laws (4.1) with control gains (k_1, k_2, k_3) = (5, 5, 7) realize the synchronization of the systems (4.1) and (4.2) as shown in Figures [13][14].

In the second numerical test the master systems has been chosen in its chaotic regime, with the parameters a = 3, b = 2.7, c = 1.7, d = 2, h = 9 (see the three scroll attractor in Fig[15]b)). Let us choose the initial conditions of the master and the slave systems respectively as (x_m(0), y_m(0), z_m(0)) = (2.5, 0, -1.5) and (x_s(0), y_s(0), z_s(0)) = (-2.5, 0.1, 1.5). Given the initial estimated parameters a_1(0) = 3, b_1(0) = 5, c_1(0) = 2, d_1(0) = 10, h_1(0) = -1 and the control gains (k_1, k_2, k_3) = (1, 2, 4) the effectiveness of the chaotic synchronization is shown in Figures [15][16].

5 The new hyperchaotic system

By introducing a feedback controller to the second equation of system (2.1), the following four-dimensional continuous-time autonomous system is obtained:

\[
\begin{align*}
\dot{x} &= y - ax + byz \\
\dot{y} &= cy - xz + z - w \\
\dot{z} &= dxz - h \\
\dot{w} &= p x + k
\end{align*}
\] (5.1)

where p and k are constants. When the parameters are chosen as:

\[
a = 36, \quad b = 0.4, \quad c = 20, \quad d = 1, \quad h = 5.07, \quad p = 5, \quad k = 0.4,
\] (5.2)

the system (5.1) is hyperchaotic, in fact the Lyapunov exponents are two positive \(\lambda_1 = 4.1927, \lambda_2 = 0.2186\), one zero \(\lambda_3 = 0.0004\) and one negative \(\lambda_4 = -25.4013\), see Fig[17] We remark that the figure shows the exponents till a time \(t = 100\) in such a way that the sign of the exponents appears much clear. However, in order to say that the system is hyperchaotic, we have run the simulation till a time \(t = 10000\) and we have found that the values of the Lyapunov exponents still maintains the same sign.

Being \(\nabla V = -a + c - h < 0\) for the chosen parameter set (5.2), the system (5.1) is dissipative and there exists a bounded hyperchaotic attractor as shown in Fig[18].

When \(k = 0\) the system admits just the origin as the unique equilibrium and it is always a saddle node as it can be straightforwardly computed.

When \(k \neq 0\) the system admits the two following equilibria:

\[
E^\pm = \left( \frac{kp}{m}, \frac{hp \pm \sqrt{\Delta}}{2bdh}, \frac{-(chp^2 - dk^2 - dhp)(hp \pm \sqrt{\Delta})}{2bdhk^2} \right),
\] (5.3)

where \(\Delta = h(4abdk^2 + hp^2)\). When the parameters are as in (5.2), the equilibria in (5.3) reduces to \(E^+ \equiv (-0.08, 161.27, -2.54, 3222.39)\) and \(E^- \equiv (-0.08, -2.83, 0.04, -56.54)\) and they are both unstable saddle nodes.

Moreover, when \(h\) varies from 5 to 10, the Lyapunov exponents spectrum is depicted in Fig[19]. One can observe that there are two positive Lyapunov exponents over quite a wide range of parameters, which implies that the system is hyperchaotic over a broad range.
6 Conclusion

In this paper we have firstly introduced the chaotic Dadras-Momeni system and its main properties and we have employed the competitive modes conjecture as a chaos diagnostic.

We have also presented two feedback controllers to regulate the chaotic system to a point. The design of these controllers is based on two methods: the optimal control and the adaptive control techniques. The proposed controllers are global, as it is shown using the Lyapunov stability theory. Each controller has its own character and can be employed in different applications, such as transportation and complex network. Moreover, we have realized, for any initial conditions, the complete synchronization of two chaotic Dadras-Momeni systems being particularly useful in secure communications, where synchronization between transmitter and receiver can be used as means of transmitting information.

Finally, motivated by the increasing interest in the study of hyperchaos, we have proposed a new hyperchaotic system obtained adding a feedback controller in the Dadras-Momeni system and we have analyzed its main properties.

The two approaches used for the control and the synchronization of the chaotic Dadras-Momeni system could be extended, with the suitable changes, to control the new hyperchaotic system. Of course, while such extension would be more computationally demanding, the methods would essentially be unchanged. The control of the higher dimensional hyperchaotic system would be the object of future research.

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References


Table 1
The initial values of co-states for different equilibrium points

<table>
<thead>
<tr>
<th>$E_i$</th>
<th>$\lambda_1(0)$</th>
<th>$\lambda_2(0)$</th>
<th>$\lambda_3(0)$</th>
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<td>$E_1$</td>
<td>$-4.8465 \times 10^{-5}$</td>
<td>$2.2256 \times 10^{-6}$</td>
<td>$3.6483 \times 10^{-5}$</td>
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<tr>
<td>$E_2$</td>
<td>$3.2437 \times 10^{-6}$</td>
<td>$2.3333 \times 10^{-5}$</td>
<td>$5.8714 \times 10^{-5}$</td>
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<td>$E_3$</td>
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<td>$-2.5000 \times 10^{-5}$</td>
<td>$1.0855 \times 10^{-5}$</td>
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<td>$E_4$</td>
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<td>$-2.3073 \times 10^{-5}$</td>
<td>$5.6080 \times 10^{-5}$</td>
</tr>
<tr>
<td>$E_5$</td>
<td>$-8.8059 \times 10^{-5}$</td>
<td>$3.3758 \times 10^{-5}$</td>
<td>$1.3009 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Figure 1: (a) Time response of the system states. (b) The chaotic attractor. The parameters are chosen as $a = 3, b = 2.7, c = 4.7, d = 2$ and $h = 9$, the initial conditions is $(5, 0, -4)^T$.

Figure 2: Plot of the mode frequencies $G_1, G_2, G_3$ for the chaotic Dadras-Momeni system with $a = 3, b = 2.7, c = 4.7, d = 2$ and $h = 9$. Note that the mode frequencies $G_1$ and $G_3$ are intermittently equal and positive, hence the modes $x(t)$ and $z(t)$ are competitive for some $t > 0$.

Figure 3: (a) The controlled behavior of the trajectories driven to the equilibrium point $E_1$. (b) The control inputs.
Figure 4: (a) The controlled behavior of the trajectories driven to the equilibrium point $E_2$. (b) The control inputs.

Figure 5: (a) The controlled behavior of the trajectories driven to the equilibrium point $E_3$. (b) The control inputs.
Figure 6: (a) The controlled behavior of the trajectories driven to the equilibrium point \( E_4 \). (b) The control inputs.

Figure 7: (a) The controlled behavior of the trajectories driven to the equilibrium point \( E_5 \). (b) The control inputs.

Figure 8: Time history of the state functions and parameter estimates for equilibrium point \( E_1 \).
Figure 9: Time history of the state functions and parameter estimates for equilibrium point \( E_2 \).

Figure 10: Time history of the state functions and parameter estimates for equilibrium point \( E_3 \).

Figure 11: Time history of the state functions and parameter estimates for equilibrium point \( E_4 \).
Figure 12: Time history of the state functions and parameter estimates for equilibrium point $E_5$.

Figure 13: Adaptive Synchronization. (a) The trajectories of the error dynamical system asymptotically converge to the origin. (b) Both the master and the slave system evolve to the period-doubling trajectory.
Figure 14: (a) The parameters of the slave system asymptotically converge to the master system parameter $a = 3, b = 2.7, c = 5.55, d = 2, h = 9$. (b) The control gains converge to zero.

Figure 15: Adaptive Synchronization. (a) The trajectories of the error dynamical system asymptotically converge to the origin. (b) Both the master and the slave system evolve to the period-doubling trajectory.
Figure 16: (a) The parameters of the slave system asymptotically converge to the master system parameter $a = 3, b = 2.7, c = 1.7, d = 2, h = 9$. (b) The control gains converge to zero.

Figure 17: The Lyapunov exponents. The parameters are specified as $a = 36, b = 0.4, c = 20, d = 1, h = 5.07, p = 5$ and $k = 0.4$. (a) One positive and one zero Lyapunov exponents. (b) One positive and one negative Lyapunov exponents.
Figure 18: The hyperchaotic attractor. The parameters are specified as $a = 36$, $b = 0.4$, $c = 20$, $d = 1$, $h = 5.07$, $p = 5$ and $k = 0.4$. The initial condition is $(2.5, 0, -2.5, -1.5)^T$. 
Figure 19: The Lyapunov exponents varying the parameter $h$. The other parameters are specified as $a = 36, b = 0.4, c = 20, d = 1, p = 5$ and $k = 0.4$. 