Global linear feedback control for the generalized Lorenz system

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Abstract

In this paper we show how the chaotic behavior of the Chen system can be controlled via feedback technique. We design both a nonlinear feedback controller and a linear one which globally regulate the closed-loop system states to a given point. We finally show that our approach works also for the whole family of the generalized Lorenz system.

Key words: Chen system, Equilibrium point stabilization, Linear control
PACS: 93B52, 93D15, 37D45, 34C28

1 Introduction

A broad class of dynamical systems modelling mechanical, electrical or chemical processes exhibits a chaotic behavior. In recent years, some of the interest in the study of chaos moved from its pure analysis to its control and utilization [10]. One of the goals of the scientific research is to design suitable and easy to implement controllers which eliminate chaos [4]. On the other hand there have been many attempts in making chaotic a nonchaotic system to exploit the created chaos in engineering applications. This procedure is called chaotification or anticontrol of chaos [5]. In [6] the chaotification of the Lorenz system in its nonchaotic regime led to the discovery of a new chaotic system, the Chen system. The Lorenz system and the Chen system have a similar mathematical structure, but they are not topologically equivalent, as the Chen system

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Preprint submitted to Chaos, Solitons and Fractals 1 July 2005
shows a more complex dynamical behavior [7]. The three equilibrium points of the Chen system in its chaotic state are all unstable and the dynamics of the system is completely irregular.

In this paper a feedback approach is proposed to drive the chaotic motion of the Chen system to a steady state. First, in the same spirit of [8], we design a nonlinear feedback controller. The corresponding linearized closed-loop system is globally regulated to any given point of the form \((x_r, x_r, b^{-1}x_r^2)\), where \(b\) is a parameter of the system. The controller we propose is very robust with respect to changes of the parameters of the system, but it has the drawback of being a nonlinear function of all the state variables of the system; this increases control implementation costs.

In the second part of the paper we follow a strategy similar to the one proposed in [1] and transform the proposed nonlinear control in a linear one. This control depends on one state variable only and maintains the same robustness and global stability properties of the nonlinear control. Therefore the results obtained in this paper can be considered an improvement of the results of [1], where the authors can achieve only local controllability, in the sense that the control designed in [1] depends on the initial data. However in [1] the authors are able to design a control that does not assume the knowledge of the equilibrium point to which the system has to be driven.

Finally, we prove that all the obtained results also hold for the generalized Lorenz system introduced in [9]. The generalized Lorenz system represents, in correspondence of the key system parameter, both the Lorenz system and the Chen system and a family of chaotic systems between them.

The plan of the paper is the following: in Section 2 we introduce the Chen system and design the global nonlinear control. In Section 3, following the strategy of [1], we make the control linear and prove that it globally stabilizes the system to the unstable points of the uncontrolled system. Finally, in Section 4 the above results are proved to hold for the all class of the generalized Lorenz systems.

## 2 Nonlinear feedback control for the Chen system

The Chen system is described by the following ordinary differential equations for the state variables \(x, y, z\) [6]:

\[
\dot{x} = a(y - x), \quad \dot{y} = bx - y - xz, \quad \dot{z} = -cz + xy
\]
\[
\begin{aligned}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x - xz + cy \\
\dot{z} &= xy - bz
\end{aligned}
\]  

(1)

where \(a, b, c\) are positive real constants.

The above system, obtained destabilizing the Lorenz system through a linear state feedback anticontrol\(\text{\textsuperscript{o}}\) [7], can be realized as an electronic circuit. The system (1) is structurally similar to the Lorenz system, being both constituted by three equations with the same kind of nonlinear terms. Moreover, they are dissipative systems so that their solutions are bounded as \(t \to \infty\). Despite of these common properties, they are not topologically equivalent: the dynamical behavior of the Chen system appears more complex than that of the Lorenz system in their own chaotic regimes and no change of coordinates can be found which maps the Chen system into the Lorenz system. If \(b(2c - a) > 0\), the system (1) has the following three equilibrium points:

\[
E_0 \equiv (0, 0, 0); \ E_\pm \equiv (\pm \sqrt{b(2c - a)}, \pm \sqrt{b(2c - a)}, 2c - a).
\]  

(2)

By choosing the values of parameters \(a = 35, b = 3, c = 28\), the above given equilibrium points are all unstable and a chaotic attractor arises for the Chen system, as shown in Fig. 1.

![Fig. 1. (a) The Chen chaotic attractor. (b) Chaotic behavior for the Chen system. The parameters are specified as \(a = 35, b = 3, c = 28\). The initial conditions are \(x(0) = -10, y(0) = 0, z(0) = 37\).](image)

In [8] a nonlinear feedback controller was proposed for the Lorenz system such that the corresponding closed-loop system can be regulated globally to any given point of the form \((x_r, x_r, b^{-1}x_r^2)\). Here we use the same feedback
technique to control the Chen system and to drag its chaotic trajectories to any point \((x_r, x_r, b^{-1}x_r^2)\).

Let us apply the external control input \(u(t)\) only in the equation for the state variable \(y\) such that the driven Chen system reads:

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x - xz + cy + u \\
\dot{z} &= xy - bz
\end{align*}
\]  

(3)

Taking the driving signal as:

\[
u = (a - c)x + xz - cy - a(y - x_r)
\]  

(4)

the Chen system is transformed into the closed-loop system:

\[
\begin{align*}
\frac{d}{dt}(x - x_r) &= -a(x - x_r) + a(y - x_r) \\
\frac{d}{dt}(y - x_r) &= -a(y - x_r) \\
\dot{z} &= xy - bz
\end{align*}
\]  

(5)

Using the same technique as in [8], it can be proved that the proposed control (4) globally stabilizes the trajectories of (1) to the point \((x_r, x_r, b^{-1}x_r^2)\). Notice that the two nontrivial unstable fixed points of the uncontrolled Chen system \(E_{\pm}\) are of the form \((x_r, x_r, b^{-1}x_r^2)\). Numerical simulations, illustrated in figure 2, show that the system trajectories are globally stabilized to the equilibrium point \(E_+\).

In (4), the parameter \(a\) represents a “measure” of the convergence rate to the point and it could be changed without losing stability. Finally, the proposed controller has very good robust properties to parameters \(a, b\) and \(c\) as it can be easily shown by a direct robustness investigation [8].

On the other hand, the controller (4) has two disadvantage. First it is nonlinear. Second to be realized it requires the knowledge of all the three state variables. All this increases the cost and the difficulties of the implementation of the controller, and makes it hardly useful in practical applications.
Fig. 2. The system is regulated to the equilibrium point $E_+(7.9373, 7.9373, 21)$. Parameters $a = 35, b = 3, c = 28$ are fixed. The controller starts to work at $t = 2$. (a) The initial conditions are $x(0) = -10, y(0) = 0, z(0) = -27$. (b) The initial conditions are $x(0) = 15, y(0) = -10, z(0) = 37$.

3 Linear feedback control

In this section we design a linear feedback driving signal which requires the knowledge of the state variable $y$ only.

Let us indicate the right hand side of the equation for $y$ in (1) as $f_2 = (c - a)x - xz + cy$. Then the controller (4) can be written as:

$$u = -f_2 - a(y - x_r).$$  \hfill (6)

Following [1], we propose an approximate feedback controller of the form:

$$u = -f_{2,e} - a(y - x_r),$$  \hfill (7)

where $f_{2,e}$ is an estimate of $f_2$. We want the approximate feedback controller (7) to converge towards the exact feedback controller (6). Introducing the estimation error $e = f_2 - f_{2,e}$, we impose to $f_{2,e}$ the following dynamics:

$$\dot{f}_{2,e} = \tau^{-1}e = \tau^{-1}(f_2 - f_{2,e}),$$  \hfill (8)

where $\tau$ is a positive parameter. From the second equation of the controlled Chen system (3) it follows that $f_2 = \dot{y} - u$. By substituting into (8), we obtain that the estimator obeys:
\[ \dot{f}_{2,e} = \tau^{-1}(y - u - f_{2,e}). \]  

(9)

This form of the estimator can not be implemented, because we assume the knowledge of the measurements of \( y \) and not of its time-derivative. For this reason we define the variable:

\[ w = \tau f_{2,e} - y \]  

(10)

representing the internal state of the estimator. The equation of the estimator (9), written in terms of the new variable \( w \) is then equivalent to:

\[ \dot{w} = -u - f_{2,e}, \quad w(0) = w_0, \]  

(11)

where:

\[ f_{2,e} = \tau^{-1}(w + y) \]  

(12)

and the feedback controller is defined by the formula (7) associated with the equation (12). This new controller is linear and its implementation requires only the measurement of the state variable \( y \). Moreover, this linear controller still regulates the trajectories of the Chen chaotic system globally to any given point with the form \((x_r, x_r, b^{-1}x_r^2)\). In fact, let us consider the governing equations of the controlled Chen system:

\[
\begin{align*}
\dot{x} &= a(y - x) \\
\dot{y} &= (c - a)x - xz + cy - \tau^{-1}(w + y) - a(y - x_r) \\
\dot{z} &= xy - bz \\
\dot{w} &= a(y - x_r)
\end{align*}
\]  

(13)

We want to prove that this system stabilizes to any given point with the form \( E(x_r, x_r, b^{-1}x_r^2, \tau[(2c - a - \tau^{-1})x_r - b^{-1}x_r^2]) \).

Let:

\[
\begin{align*}
\xi_1 &= x - x_r; \quad \xi_2 = y - x_r; \quad \xi_3 = z - b^{-1}x_r^2; \\
\xi_4 &= w - \tau((2c - a - \tau^{-1})x_r - b^{-1}x_r^2).
\end{align*}
\]
be the coordinates transformation which translates the point $E$ into the origin. Then, the system (13) becomes:

$$
\begin{align*}
\dot{\xi}_1 &= a(\xi_2 - \xi_1) \\
\dot{\xi}_2 &= (c - a)(\xi_1 + \xi_2) - \xi_1 \xi_3 - x_r \xi_3 - b^{-1} x_r^2 \xi_1 - \tau^{-1}(\xi_4 + \xi_2) \\
\dot{\xi}_3 &= x_r(\xi_1 + \xi_2) + \xi_1 \xi_2 - b \xi_3 \\
\dot{\xi}_4 &= a \xi_2
\end{align*}
$$

(14)

We define a Lyapunov function $V : \mathbb{R}^4 \to \mathbb{R}$ as:

$$
V = \frac{a - c + b^{-1} x_r^2}{a} \xi_1^2 + \xi_2^2 + \xi_3^2 + \frac{1}{a \tau} \xi_4^2.
$$

(15)

Since $\tau > 0$ and when $a - c > 0$ (which is the case, for example, when $a = 35$, $b = 3$ and $c = 28$), the function $V$ is positive definite. Its time-derivative along the system trajectories is:

$$
\dot{V} = - \left( \left( a - c + \frac{3}{4} x_r^2 \right) \xi_1^2 + (a - c + \tau^{-1}) \xi_2^2 + b \left( \xi_3 - \frac{x_r}{2b} \xi_1 \right)^2 \right).
$$

(16)

The function $\dot{V}$ when $a - c > 0$ is less or equal to zero for all $\xi \equiv (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$. Moreover, it can be easily proved that the set:

$$
S = \{ \xi \in \mathbb{R}^4 : \dot{V} = 0 \} = \{(0, 0, 0, \xi_4), \xi_4 \in \mathbb{R}\}
$$

(17)

does not contain any nontrivial system trajectories. Then, the hypotheses of the Krasovskii-La Salle invariance principle [11] are satisfied $\forall \tau > 0$ and $\forall x_r \in \mathbb{R}$. We have thus proved the following:

**Theorem 1** Suppose $a - c > 0$. Then the control (7), together with (11) and (12), globally regulates the system (3) to any given point of the form $(x_r, x_r, b^{-1} x_r^2), \forall \tau > 0$. The dynamics of the controlled system is ruled by (13).

This improves the result in [1] in the sense that the proposed control works globally and independently on the value of $\tau$.

Numerical simulations show how the state variables $x, y$ and $z$ of the Chen controlled system (13) are regularized to the equilibrium point $E_+(7.9373, 7.9373, 21)$.

7
Notice that the choice of the parameter $\tau$ only influences the rate of convergence to the point $(x_\tau, x_\tau, b^{-1}x_\tau^2)$ (see Fig. 3.b).

![Graph](image1)

(a) ![Graph](image2)

(b) ![Graph](image3)

(c)

Fig. 3. The system is regulated to the equilibrium point $E_+(7.9373, 7.9373, 21)$. $a = 35, b = 3, c = 28; x(0) = 15, y(0) = -10, z(0) = 37$. The controller starts to work at $t = 2$. The point stabilization for the Chen system also holds independently on initial conditions of $w$: while $w(0)$ is chosen equal to $\tau f_2(0) - y(0)$ in numerical simulations showed in figures (a) and (b), it is completely arbitrary in figure (c). (a) $\tau = 0.1$; (b) $\tau = 10$; (c) $\tau = 100$, $w(0) = 100$.

4 Linear control for a generalized Lorenz system

The so-called generalized Lorenz system, containing both the Lorenz and the Chen system, has been introduced in [2] and a corresponding generalized Lorenz canonical form has been recently discovered in [3]. All the systems belonging to these class are structurally similar: they are dissipative and have the same kind of equilibria stability, but they are not topologically equivalent. Moreover, some of the systems in this class represent a sort of transition be-
tween the Lorenz and the Chen system. To analyze in a unified system which depends on one parameter only, both the Lorenz and the Chen system and all the chaotic systems between them, the following system:

\[
\begin{align*}
\dot{x} &= (25\alpha + 10)(y - x) \\
\dot{y} &= (28 - 35\alpha)x - xz + (29\alpha - 1)y \\
\dot{z} &= xy - \frac{\alpha + 8}{3}z
\end{align*}
\]  
(18)

has been introduced in [9], where \( \alpha \in [0, 1] \).

This system is chaotic \( \forall \alpha \in [0, 1] \). In particular, it reduces to the Lorenz system for \( \alpha = 0 \) and to the Chen system for \( \alpha = 1 \), both in their chaotic regime. Moreover, as the parameter \( \alpha \) increases from 0 to 1, the chaotic dynamics of the system (18) evolves from the Lorenz attractor to the Chen attractor. The system (18) has the following three equilibrium points:

\[
E_0 \equiv (0, 0, 0); \quad E_{\pm} \equiv (\pm \sqrt{(8 + \alpha)(9 - 2\alpha)}, \pm \sqrt{(8 + \alpha)(9 - 2\alpha)}, 27 - 6\alpha)
\]

and they are all unstable for all \( \alpha \in [0, 1] \).

In what follows we shall show that the results presented in this paper can be extended to the system (18). The system (18) is regulated to any given point of the form \( (x_r, x_r, 3x_r^2/(\alpha + 8)) \) by designing a linear controller as in section 3. The driving signal \( u \) is given by:

\[
u = -f_{2,e} - \gamma(y - x_r),
\]
(19)

where \( \gamma = 112 \) and \( f_{2,e} \) is an estimate of \( f_2 = (28 - 35\alpha)x - xz + (29\alpha - 1)y \), which obeys to equation (9). The governing equations of the controlled system (18) in the second state variable are:

\[
\begin{align*}
\dot{x} &= (25\alpha + 10)(y - x) \\
\dot{y} &= (28 - 35\alpha)x - xz + (29\alpha - 1)y - \tau^{-1}(w + y) - \gamma(y - x_r) \\
\dot{z} &= xy - \frac{\alpha + 8}{3}z \\
\dot{w} &= \gamma(y - x_r)
\end{align*}
\]
(20)

where \( w \) is defined in (10).
The proof that this driven system regulates to any given point with the form
\((x_r, x_r, 3x_r^2/(\alpha + 8), \tau[(-6\alpha + 27)x_r - 3x_r^3/(\alpha + 8)] - x_r)\) follows the same steps as in the proof of Theorem 1.

One imposes the following change of variables:

\[
\begin{align*}
\xi_1 &= x - x_r; \quad \xi_2 = y - x_r; \quad \xi_3 = z - \frac{3}{\alpha + 8}x_r^2; \\
\xi_4 &= w - \tau \left((-6\alpha + 27)x_r - \frac{3}{\alpha + 8}x_r^3\right) + x_r,
\end{align*}
\]

so that the system (20) becomes:

\[
\begin{aligned}
\dot{\xi}_1 &= (25\alpha + 10)(\xi_2 - \xi_1) \\
\dot{\xi}_2 &= (28 - 35\alpha)\xi_1 + (4\alpha - 11)\xi_2 - \xi_1\xi_3 - x_r\xi_3 - \frac{3}{\alpha + 8}x_r^2\xi_1 - \tau^{-1}(\xi_4 + \xi_2) \\
\dot{\xi}_3 &= \xi_1\xi_2 - \frac{\alpha + 8}{3}\xi_3 + x_r(\xi_1 + \xi_2) \\
\dot{\xi}_4 &= \gamma\xi_2
\end{aligned}
\]

The Lyapunov function \(V : \mathbb{R}^4 \to \mathbb{R}\) is given by:

\[
V = (25\alpha + 10)^{-1}\left(\frac{3}{\alpha + 8}x_r^2 + 35\alpha + \sigma - 28\right)\xi_1^2 + \xi_2^2 + \xi_3^2 + \frac{1}{\tau\gamma}\xi_4^2,
\]

where we assume \(\sigma = 0\) for all \(\alpha \in \left(\frac{4}{5}, 1\right]\) and \(\sigma > 28\) for all \(\alpha \in \left[0, \frac{4}{5}\right]\). It is positive definite \(\forall \tau > 0\). Moreover:

\[
\dot{V} = -\left[\frac{9}{4\alpha + 8}x_r^2\xi_1^2 + \left(1 + \gamma - 29\alpha - \frac{\sigma^2}{35\alpha + \sigma - 28} + \tau^{-1}\right)\xi_2^2 + (35\alpha + \sigma - 28)\left(\xi_1 - \frac{\sigma}{35\alpha + \sigma - 28}\xi_2\right)^2 + \frac{\alpha + 8}{3}\left(\xi_3 - \frac{1}{2}x_r\frac{3}{\alpha + 8}\xi_1\right)^2\right]
\]

is less or equal to zero, by choosing \(\sigma = 0\) for all \(\alpha \in \left(\frac{4}{5}, 1\right]\) and \(28 < \sigma < \frac{(1 + \gamma - 29\alpha) + \sqrt{(29\alpha - 1 - \gamma)(111 - 111\alpha - \gamma)}}{2}\) for all \(\alpha \in \left[0, \frac{4}{5}\right]\).

Furthermore, the set:
contains only trivial trajectories, then according to the Krasovskii-La Salle invariance principle, the origin is a globally asymptotically stable point \( \forall \tau > 0 \) and \( \forall \alpha \in [0,1] \) of the system (21). Then, the system (20) is globally regulated to any given point of the form \((x_r, x_r, 3x_r^2/(\alpha + 8), \tau[(-6\alpha + 27)x_r - 3x_r^3/(\alpha + 8)] - x_r), \forall \tau > 0\).

Numerical simulations illustrating the regularization of the system (18) to the equilibrium point \( E_+ \equiv (\sqrt{(8 + \alpha)(9 - 2\alpha)}, \sqrt{(8 + \alpha)(9 - 2\alpha)}, 27 - 6\alpha) \) are shown in Figures 4-7 at some values of \( \alpha \in [0,1] \).

Fig. 4. (a) Lorenz attractor, \( \alpha = 0 \). (b) Regulated system to the point \( E_+(8.4853, 8.4853, 27); \tau = 10 \). The controller starts to work at \( t = 2 \).

Fig. 5. (a) Variant of the Lorenz attractor, \( \alpha = 0.4 \). (b) Regulated system to the point \( E_+(8.2994, 8.2994, 24.6); \tau = 10 \). The controller starts to work at \( t = 2 \).
Fig. 6. (a) Variant of the Chen attractor, $\alpha = 0.5$. (b) Regulated system to the point $E_+(8.2462, 8.2462, 24); \tau = 10$. The controller starts to work at $t = 2$.

Fig. 7. (a) Chen attractor, $\alpha = 1$. (b) Regulated system to the point $E_+(7.9373, 7.9373, 21); \tau = 10$. The controller starts to work at $t = 2$.

5 Conclusions

In this paper feedback techniques are used to control the Chen system in its chaotic regime and to drag its trajectories to a point. The global stabilization of the chaotic Chen system is first achieved by using a nonlinear controller. Then a linear controller is designed which preserves the property of robustness of the nonlinear one and still globally regulates the system to one of its unstable points. The linear controller depends on one parameter $\tau$, which only influences the rate of convergence to the point. The global stabilization is achieved for all the values of $\tau$. Moreover the proposed linear controller requires the exact knowledge of one of the state variables only, which makes it easy to implement. Finally, all the results are successfully applied to a class of generalized Lorenz system which contains both the Lorenz and the Chen system and a family of...
chaotic system between them.

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