Voronoi cells of beta-integers

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Beta-numeration (Rényi, Parry)

$\beta > 1$, non-integer number.

β-expansion of $x > 0$: greedy algorithm

$\beta^k \leqslant x < \beta^{k+1}$

$x_k = \lfloor x/\beta^k \rfloor$ and $r_k = \{x/\beta^k\}$.

For $i < k$, let $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$.

$$\langle x \rangle_\beta = x_k x_{k-1} \dots x_1 x_0 \cdot x_{-1} x_{-2} \dots$$

$x_i \in \mathbb{B} = \{0, \dots, \lfloor \beta \rfloor\}$.

β-expansion of 1: $d_\beta(1) = (t_i)_{i \geqslant 1}$

$T_\beta(x) = \beta x \bmod 1$ and $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$

Beta-integers

The set of *β-integers* is

$$\mathbb{Z}_\beta \quad = \quad \left\{ x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_N x_{N-1} \cdots x_1 x_0 \right\}.$$

The set $\mathbb{Z}_\beta^+$, of positive β-integers, is ordered by the lexicographical order on β-expansions.
Its n-th element is called b_n.

By symmetry

$$b_{-n} = -b_n$$

Example $\varphi = \frac{1+\sqrt{5}}{2}$ $\quad d_\varphi(1) = 11$

$$
\begin{aligned}
b_0 &= 0 & \langle b_0 \rangle_\varphi &= 0 \\
b_1 &= 1 & \langle b_1 \rangle_\varphi &= 1 \\
b_2 &= \varphi & \langle b_2 \rangle_\varphi &= 10 \\
b_3 &= \varphi^2 & \langle b_3 \rangle_\varphi &= 100 \\
b_4 &= \varphi^2 + 1 & \langle b_4 \rangle_\varphi &= 101
\end{aligned}
$$

Pisot number : algebraic integer $\beta > 1$ such that all its Galois conjugates are < 1 in modulus.

The golden mean $\varphi = \frac{1+\sqrt{5}}{2}$ is a Pisot number.

If β is a Pisot number then $d_\beta(1)$ is finite or eventually periodic (Bertrand, Schmidt).

β is a Parry number if $d_\beta(1)$ finite or eventually periodic.

Substitution tilings

To a Parry number β is associated a substitution σ_β which defines a tiling of the positive real line with a finite number of tiles.

1. $d_\beta(1) = t_1 \cdots t_m$ is finite

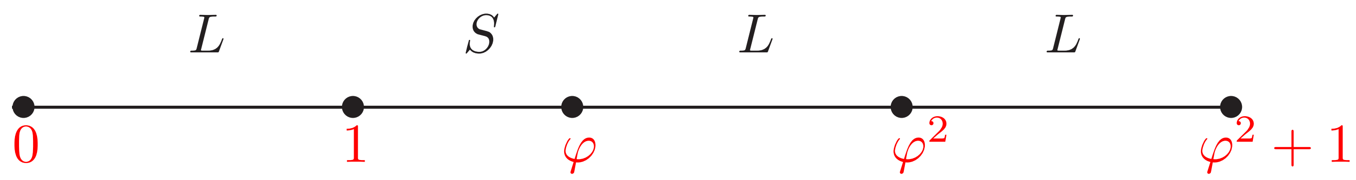
$$\sigma_\beta = \begin{cases} a_0 \mapsto a_0^{t_1} a_1 \\ a_1 \mapsto a_0^{t_2} a_2 \\ \vdots \\ a_{m-2} \mapsto a_0^{t_{m-1}} a_{m-1} \\ a_{m-1} \mapsto a_0^{t_m} \end{cases}$$

Fixed point $u_\beta = \sigma_\beta^\infty(a_0)$

The vertices of the tiling are labelled by the positive β-integers.

Example Fibonacci word. $\varphi = \frac{1+\sqrt{5}}{2}$

$$\sigma_\varphi = \begin{cases} L \mapsto LS \\ S \mapsto L \end{cases}$$



2. $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$ is infinite eventually periodic

$$\sigma_\beta = \begin{cases} a_0 \mapsto a_0^{t_1} a_1 \\ a_1 \mapsto a_0^{t_2} a_2 \\ \vdots \\ a_{m+p-2} \mapsto a_0^{t_{m+p-1}} a_{m+p-1} \\ a_{m-1} \mapsto a_0^{t_{m+p}} a_m \end{cases}$$

Fixed point $u_\beta = \sigma_\beta^\infty(a_0)$

Quasicrystallography

Crystals: solids in dimension 2 or 3, with atoms arranged periodically. Symmetry of order n.
n must satisfy

$$2\cos\frac{2\pi}{n} \in \mathbb{Z}$$

hence $n = 2, 3, 4, 6$.

Quasicrystal Alloy aluminium-manganese with order 5 symmmetry Shechtman and al. 1984
Quasi-periodicity.

Cyclotomic Pisot number β such that

$$\mathbb{Z}[2\cos\frac{2\pi}{n}] = \mathbb{Z}[\beta]$$

$\mathbb{Z}[\beta] + \mathbb{Z}[\beta]\exp(2i\pi/n)$ is a ring invariant under rotation of order n.

$\rho = 2\cos\frac{2\pi}{n}$

Quasicrystals in the real world

• $n = 5$ or $n = 10$: $\beta = \rho = \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}$,
$M_\beta(X) = X^2 - X - 1$

• $n = 8$: $\beta = 1 + \rho = 1 + \sqrt{2} = 1 + 2\cos\frac{\pi}{4}$,
$M_\beta(X) = X^2 - 2X - 1$

• $n = 12$: $\beta = 2 + \rho = 2 + \sqrt{3} = 2 + 2\cos\frac{\pi}{6}$,
$M_\beta(X) = X^2 - 4X + 1$.

They are quadratic Pisot units

Other cyclotomic Pisot number

• $n = 7$ or $n = 14$: $\beta = 1 + \rho = 1 + 2\cos\frac{\pi}{7}$,
$M_\beta(X) = X^3 - 2X^2 - X + 1$

Quadratic Pisot units

$\beta > 1$ root of $X^2 - aX - 1$, $a \geqslant 1$.

Canonical alphabet $\mathbb{B} = \{0, 1, \ldots, a\}$

The β-expansion of 1 is finite $d_\beta(1) = a1$

The substitution σ_β is defined on the alphabet $\mathbb{A} = \{L, S\}$ by

$$\sigma_\beta = \begin{cases} L \mapsto L^a S \\ S \mapsto L\,. \end{cases}$$

To each letter of $\mathbb{A}$ we associate a tile of length $\ell(L) = 1$, and $\ell(S) = T_\beta(1) = \beta - a = 1/\beta$.

Fixed point $\sigma_\beta^\infty(L)$ is a Sturmian word.

$\beta > 1$ root of $X^2 - aX + 1$, $a \geqslant 3$.

Canonical alphabet $\mathbb{B} = \{0, 1, \ldots, a-1\}$

The β-expansion of 1 is eventually periodic equal to $d_\beta(1) = (a-1)(a-2)^\omega$.

The substitution σ_β is defined on the alphabet $\mathbb{A} = \{L, S\}$ by

$$\sigma_\beta = \begin{cases} L \mapsto L^{a-1}S \\ S \mapsto L^{a-2}S\,. \end{cases}$$

$\ell(L) = 1$, and

$\ell(S) = T_\beta(1) = \beta - (a-1) = 1 - 1/\beta$.

Fixed point $\sigma_\beta^\infty(L)$ is a Sturmian word.

Cubic Pisot units

Tribonacci: β root of $X^3 - X^2 - X - 1$

Canonical alphabet is $\mathbb{B} = \{0, 1\}$
$d_\beta(1) = 111.$
The substitution σ_β is defined on the alphabet $\mathbb{A} = \{L, M, S\}$ by

$$\sigma_\beta = \begin{cases} L \mapsto LM \\ M \mapsto LS \\ S \mapsto L\,. \end{cases}$$

$\ell(L) = 1,$
$\ell(M) = T_\beta(1) = \beta - 1,$
$\ell(S) = T_\beta^2(1) = \beta^2 - \beta - 1.$

Fixed point $\sigma_\beta^\infty(L)$ is a Arnoux-Rauzy word:
Complexity $\mathcal{C}(n) = 2n + 1$
One left and one right special factor of each length with 3 extensions

β root of $X^3 - 2X^2 - X + 1$

Cyclotomic Pisot unit with a 7-fold symmetry

Canonical alphabet is $\mathbb{B} = \{0, 1, 2\}$
$d_\beta(1) = 2(01)^\omega$
The substitution σ_β is defined on the alphabet
$\mathbb{A} = \{L, M, S\}$ by

$$\sigma_\beta = \begin{cases} L \mapsto LLS \\ S \mapsto M \\ M \mapsto LS \end{cases}$$

$\ell(L) = 1,$
$\ell(S) = T_\beta(1) = \beta - 2,$
$\ell(M) = T_\beta^2(1) = \beta^2 - 2\beta.$

Fixed point $\sigma_\beta^\infty(L)$ is a not an Arnoux-Rauzy word, but complexity $\mathcal{C}(n) = 2n + 1$.

Meyer sets and Voronoi cells

Voronoi cell $\mathcal{V}(\lambda)$ of $\lambda \in \Lambda$ discrete set in $\mathbb{R}^n$

$$\mathcal{V}(\lambda) = \{x \in \mathbb{R}^n \mid d(x-\lambda) \leqslant d(x-\lambda'),\ \lambda' \in \Lambda\},$$

where d is the Euclidean distance in $\mathbb{R}^n$.

The set of Voronoi cells of a discrete set Λ forms a tiling of $\mathbb{R}^n$ called the Voronoi tessellation of $\mathbb{R}^n$ induced by Λ.

If Λ is a Meyer set, its Voronoi tessellation contains a finite number of tiles (Lagarias).

If β is a Pisot number, then the set $\mathbb{Z}_\beta$ of β-integers is a Meyer set (Burdik, Frougny, Gazeau, Krejcar).

Windows

Quadratic Pisot units

$\beta > 1$ a quadratic Pisot unit and β' the other root.

Galois conjugation automorphism is the map

$$x = \sum_{k \leqslant N} x_k \beta^k \mapsto x' = \sum_{k \leqslant N} x_k {\beta'}^k$$

Window of positive β-integers is Ω

$$\Omega = \overline{\{x' \mid x \in \mathbb{Z}_\beta^+\}}\,.$$

$x \in \mathbb{Z}[\beta] \cap \mathbb{R}^+$ is a positive β-integer if and only if its conjugate x' belongs to the window
$\Omega = (-1, \beta)$ if $\beta^2 = a\beta + 1$,
$\Omega = (0, \beta)$ if $\beta^2 = a\beta - 1$ (Burdik, Frougny, Gazeau, Krejcar)

Cubic Pisot units

$\beta > 1$ the Tribonacci number, and α and α^c its Galois conjugates.

Galois conjugation automorphism is the map

$$x = \sum_{k \leqslant N} x_k \beta^k \mapsto x' = \sum_{k \leqslant N} x_k \alpha^k$$

Window $\Omega = \overline{\{x' \mid x \in \mathbb{Z}_\beta^+\}}$ is a compact subset of $\mathbb{C}$ with a fractal boundary, the Rauzy fractal.

$\beta > 1$ root of $X^3 - 2X^2 - X + 1$.

The other roots are the real numbers $\alpha_1 = \beta^2 - 2\beta$ and $\alpha_2 = -\beta^2 + \beta + 2$.

Galois conjugation automorphism is the map

$$x = \sum_{k \leqslant N} x_k \beta^k \mapsto x' = \sum_{k \leqslant N} x_k (\alpha_1^k + \alpha_2^k e^{i4\pi/7})$$

The window Ω of positive β-integers is $\Omega = \overline{\{x' \mid x \in \mathbb{Z}_\beta^+\}}$.

The determination of the window of positive β-integers is an open problem.

Beta-integers Voronoi cells

If a β-integer is the common vertex of the tiles A and B, it is said to be an AB β-integer, and its Voronoi cell is said to be an AB Voronoi cell.

The window associated with positive AB β-integers is

$$\Omega_{AB} = \overline{\{x' \mid x \in \mathbb{Z}_\beta^+,\ x \text{ is } AB\}}\,.$$

$\beta^2 = a\beta + 1$, $a \geqslant 1$. $d_\beta(1) = a1$.

PROPOSITION **1** . *b_n a positive β-integer.*

1. b_n is SL $\iff$ $\langle b_n \rangle_\beta$ ends by an odd number of 0's $\iff$ $b'_n \in \Omega_{SL} = (-1, 0)$.

2. b_n is LL $\iff$ for $n \geqslant 1$, $\langle b_n \rangle_\beta$ ends by either an even number of 0's, or by $h \in \{1, \ldots, a-1\}$ $\iff$ $b'_n \in \Omega_{LL} = [0, \beta - 1)$.

3. b_n is LS $\iff$ $\langle b_n \rangle_\beta$ ends by a $\iff$ $b'_n \in \Omega_{LS} = (\beta - 1, \beta)$.

Partition of the window of LL β-integers as

$$\Omega_{LL} = \bigcup_{0 \leqslant h \leqslant a-1} \Omega_{LL}(h)\,,$$

where $\Omega_{LL}(h)$ is the window associated with positive LL β-integers such that their β-expansion ends by $h \in \{0, \cdots, a-1\}$.

Proposition **2** . *b_n a LL β-integer, then $\langle b_n \rangle_\beta$ ends by*

1. an even number of 0's $\Longleftrightarrow$
$b'_n \in \Omega_{LL}(0) = [0, \frac{1}{\beta})$

2. an $h \in \{1, \ldots, a-1\}$ $\Longleftrightarrow$
$b'_n \in \Omega_{LL}(h) = (\frac{1}{\beta} + h - 1, \frac{1}{\beta} + h)$.

Remark: $\frac{1}{\beta} = \beta - a$

Example $\gamma = 1 + \sqrt{2}$, $d_\gamma(1) = 21$

$$\sigma_\gamma = \begin{cases} L \mapsto LLS \\ S \mapsto L \end{cases}$$

$u_\gamma = LLSLLSLLLSLLS \cdots$

γ-exp.	γ-int.	type	window
1	1	LL	$(\gamma - 2, \gamma - 1)$
2	2	LS	$(\gamma - 1, \gamma)$
10	γ	SL	$(-1, 0)$
11	$\gamma + 1$	LL	$(\gamma - 2, \gamma - 1)$
12	$\gamma + 2$	LS	$(\gamma - 1, \gamma)$
20	2γ	SL	$(-1, 0)$
100	γ^2	LL	$[0, \gamma - 2)$
101	$\gamma^2 + 1$	LL	$(\gamma - 2, \gamma - 1)$
102	$\gamma^2 + 2$	LS	$(\gamma - 1, \gamma)$
110	$\gamma^2 + \gamma$	SL	$(-1, 0)$

$\beta^2 = a\beta - 1$, $a \geqslant 3$. $d_\beta(1) = (a-1)(a-2)^\omega$.

$\mathcal{M} = (a-1)(a-2)^*$ set of maximal words in the lexicographical order.

PROPOSITION **3** . *b_n a positive β-integer, $n \geqslant 1$.*

1. b_n is SL $\iff$ $\langle b_n \rangle_\beta$ ends by 0 $\iff$ $b'_n \in \Omega_{SL} = (0,1)$.

2. b_n is LL $\iff$ $\langle b_n \rangle_\beta$ ends by a word $w \notin \mathcal{M} \cup 0$ $\iff$ $b'_n \in \Omega_{LL} = [1, \beta - 1)$.

3. b_n is LS $\iff$ $\langle b_n \rangle_\beta$ ends by a word $w \in \mathcal{M}$ $\iff$ $b'_n \in \Omega_{LS} = (\beta - 1, \beta)$.

$\beta^2 = a\beta - 1,\ a \geqslant 3.$

PROPOSITION **4** . *Let* b_n *be LL then* $\langle b_n \rangle_\beta$ *ends by*

1. *an* $h \in \{1, \ldots, a-3\} \iff$
$b'_n \in \Omega_{LL}(h) = [h, h+1)$

2. $(a-2)$ *not prefixed by an element of* $\mathcal{M} \iff$
$b'_n \in \Omega_{LL}(a-2) = (a-2, \beta - 1).$

Tribonacci $\beta^3 = \beta^2 + \beta + 1$. $d_\beta(1) = 111$.

PROPOSITION **5** . *b_n a positive β-integer, $n \geqslant 1$.*

1. b_n is LM $\iff$ $\langle b_n \rangle_\beta$ ends by 01, or $n = 1$ and $\langle b_1 \rangle_\beta = 1$.

2. b_n is LS $\iff$ $\langle b_n \rangle_\beta$ ends by 011, or $n = 3$ and $\langle b_3 \rangle_\beta = 11$.

3. b_n is ML $\iff$ $\langle b_n \rangle_\beta$ ends by $10(000)^q$, $q \in \mathbb{N}$.

4.b_n is SL $\iff$ $\langle b_n \rangle_\beta$ ends by $100(000)^q$, $q \in \mathbb{N}$.

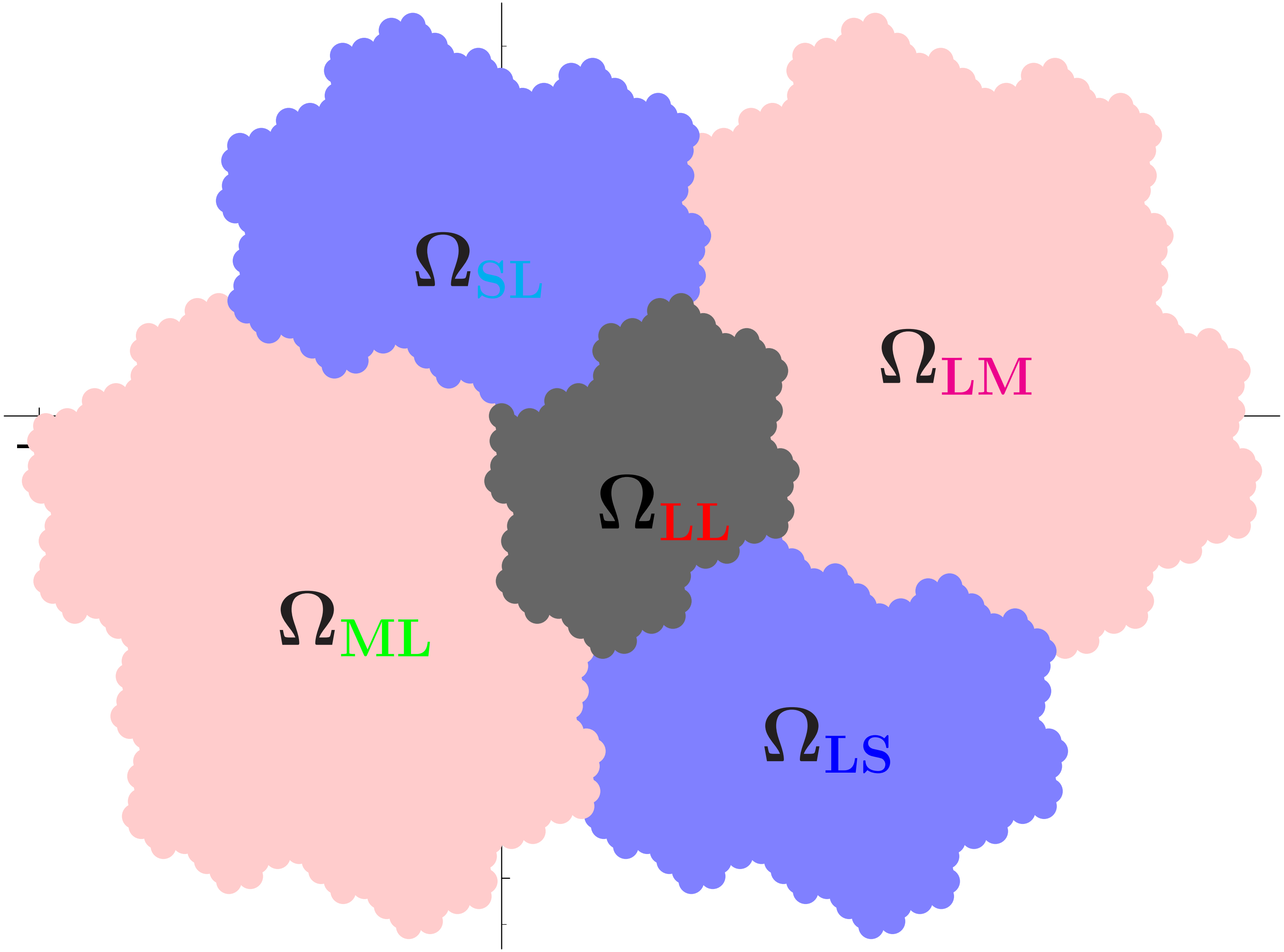
5. b_n is LL $\iff$ $\langle b_n \rangle_\beta$ ends by $1000(000)^q$, $q \in \mathbb{N}$.

$\beta^3 = \beta^2 + \beta + 1$ Tribonacci

$LMLSLMLLMLSLMLMLS\cdots$

β-exp.	type
1	LM
10	ML
11	LS
100	SL
101	LM
110	ML
1000	LL
1001	LM
1010	ML
1011	LS
1100	SL
1101	LM
10(000)	ML

$\Omega = \overline{(\mathbb{Z}_\beta)'} = \text{Rauzy fractal}$



Usually the Rauzy fractal is divided into three basic tiles T_0, T_{01} and T_{011}

$$T_0 = \Omega_{SL} \cup \Omega_{ML} \cup \Omega_{LL}$$

$$T_{01} = \Omega_{LM}$$

$$T_{011} = \Omega_{LS}$$

Domain exchange ρ on the Rauzy fractal

$$T_0 = \Omega_{SL} \cup \Omega_{ML} \cup \Omega_{LL} \overset{\rho}{\mapsto} \Omega_{LS} \cup \Omega_{LM} \cup \Omega_{LL}$$

$$T_{01} = \Omega_{LM} \overset{\rho}{\mapsto} \Omega_{ML}$$

$$T_{011} = \Omega_{LS} \overset{\rho}{\mapsto} \Omega_{SL}$$

PROPOSITION **6** . *In the Rauzy fractal we have the following relations*

(i) $\Omega_{ML} = \Omega_{LM} + \alpha^{-1} + \alpha^{-2}$

(ii) $\Omega_{SL} = \Omega_{LS} + \alpha^{-1}$

(iii) $\Omega_{LL} = \alpha\Omega_{LS} + 1 = \alpha^2\Omega_{LM} + \alpha + 1.$

$\beta^3 = 2\beta^2 + \beta - 1$. $d_\beta(1) = 2(01)^\omega$.

$\mathcal{M}_1 = 2(01)^*$ and $\mathcal{M}_2 = 2(01)^*0$.

$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ is the set of maximal words in the lexicographical order.

Proposition 7 . *b_n a positive β-integer, $n \geqslant 1$.*

1. *b_n is LL $\Longleftrightarrow$ $\langle b_n \rangle_\beta$ ends by $w1$ where $w \notin \mathcal{M}_2$.*
2. *b_n is LS $\Longleftrightarrow$ $\langle b_n \rangle_\beta$ ends by a word $w \in \mathcal{M}_1$.*
3. *b_n is SL $\Longleftrightarrow$ $\langle b_n \rangle_\beta$ ends by $w0$ where $w \notin \mathcal{M}_1$ or by $(0)^{2q+1}$, $q \in \mathbb{N}^*$.*
4. *b_n is SM $\Longleftrightarrow$ $\langle b_n \rangle_\beta$ ends by a word $w \in \mathcal{M}_2$.*
5. *b_n is ML $\Longleftrightarrow$ $\langle b_n \rangle_\beta$ ends by $(0)^{2q+2}$, $q \in \mathbb{N}$.*