The Inclusion Problem for Unambiguous Rational Trace Languages

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Outline

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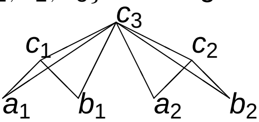
Introduction

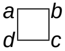
- Trace Languages (Mazurkiewicz '77), subsets of free partially commutative monoids.
- Inclusion Problem for $\mathcal{C}$:

$$\text{Given } L_1, L_2 \in \mathcal{C},\ L_1 \subseteq L_2?$$

- Our result:
  Inclusion is decidable for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ with $\Sigma = \{a_1, b_1, c_1, a_2, b_2, c_2, c_3\}$ and C given by



Known Results

- Inclusion for $\mathrm{Rat}(\Sigma, C)$ is decidable when C is transitive (Bertoni '85).
- Inclusion for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ is undecidable if C contains 

(Bertoni, Goldwurm, Mauri, Sabadini '95).
- Inclusion for $\mathrm{Rat}_{\mathrm{Fin}}(\Sigma, C)$ is decidable if $\Sigma = \Sigma_1 \dot\cup \Sigma_2$ and $C = (\Sigma_1 \times \Sigma \cup \Sigma \times \Sigma_1) \setminus I$ (Bertoni, Massazza '98).
- Equivalence for $\mathrm{Rat}(\Sigma, C)$ is undecidable when $C = {}_a\!\bigwedge^{b}\!{}_c$ (Gibbons and Rytter '86, Aalbersberg and Hoogeboom '89).
- Equivalence for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ is decidable for any C (Varricchio '90).

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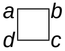
- Inclusion for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ is undecidable if C contains 

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Trace Languages

We need:

- a finite alphabet $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$,
- a commutation relation $C \subseteq \Sigma \times \Sigma$,
- ρ_C (the congruence generated by C),
- the free partially commutative monoid $F(\Sigma, C) = \Sigma^\star / \rho_C$.

Definition (Trace)

A trace is an element of $F(\Sigma, C)$, i.e. an equivalence class $[w]_{\rho_C}$ generated by $w \in \Sigma^\star$.

Definition (Trace Language)

A trace language is a subset of $F(\Sigma, C)$.

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Formal series

Definition (Formal Series)

Given a monoid M, $\psi : M \mapsto \mathbb{Q}$, $\psi = \sum_{m \in M} (\psi, m)m$.

Definition (Characteristic Series)

Given $L \subseteq M$, $\chi_L = \sum_{m \in L} 1m$.

Definition (Generating Function)

Given a formal series ϕ on M, $f_\phi(z) = \sum_{m \in M} (\phi, m) z^{|m|}$.

The generating function of $L \subseteq F(\Sigma, C)$ is

$$f_L(z) = f_{\chi_L}(z) = \sum_{t \in F(\Sigma,C)} (\chi_L, t) z^{|t|} = \sum_{n \geq 0} c_n z^n, \ (c_n = \sharp\{t \in L \mid |t| = n\})$$

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Operations on Formal Series

We consider $\mathbf{Q}\langle\langle\Sigma\rangle\rangle$ equipped with

- Sum: $(\phi + \psi, w) = (\phi, w) + (\psi, w)$,
- Cauchy product: $(\phi \cdot \psi, w) = \sum_{xy=w} (\phi, x) \cdot (\psi, y)$,
- Hadamard product: $(\phi \odot \psi, w) = (\phi, w) \cdot (\psi, w)$.

Moreover, on $\mathbf{Q}[[\Sigma]]$ we consider also

- Partial derivative:
  $(\partial_{\sigma_i}\phi, \sigma_1^{a_1} \cdots \sigma_i^{a_i} \cdots \sigma_n^{a_n}) = (a_i + 1)(\phi, \sigma_1^{a_1} \cdots \sigma_i^{a_i+1} \cdots \sigma_n^{a_n})$,
- Primitive diagonal:

$$(\Delta_{\sigma_i\sigma_j}\phi, \sigma_1^{a_1} \cdots \sigma_{j-1}^{a_{j-1}} \sigma_{j+1}^{a_{j+1}} \cdots \sigma_n^{a_n}) = (\phi, \sigma_1^{a_1} \cdots \sigma_{j-1}^{a_{j-1}} \sigma_j^{a_i} \sigma_{j+1}^{a_{j+1}} \cdots \sigma_n^{a_n}),$$

- Sustitution: $\phi(\psi_1(\tau_1, \ldots, \tau_m), \ldots, \psi_n(\tau_1, \ldots, \tau_m))$.

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Rational Series and Languages

Definition (Rational series on $F(\Sigma, C)$)

ϕ : F$(\Sigma, C) \mapsto \mathbb{Q}$ is rational iff $\exists n \geq 1$, $\mu : \Sigma^\star \mapsto \mathbb{Q}^{n \times n}$ and $\lambda \in \mathbb{Q}^{1 \times n}, \gamma \in \mathbb{Q}^{1 \times n}$ s.t.

$$(\phi, t) = \sum_{\substack{w \in \Sigma^\star \\ [w]_C = t}} \lambda \mu w \gamma \ .$$

Definition (Unambiguous Rational Trace Languages)

$L \subseteq F(\Sigma, C)$ is unambiguous rational iff χ_L is rational.

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Our approach to the Inclusion Problem

Fact

$L_1 \subseteq L_2$ iff $f_{L_1}(z) = f_{L_1 \cap L_2}(z)$, that is, iff

$$f_{\chi_{L_1}}(z) = f_{\chi_{L_1} \odot \chi_{L_2}}(z).$$

The Algorithm:

Step 1: Given $L_1, L_2 \in \mathrm{Rat_U}(\Sigma, C)$ compute $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$, $f_{\chi_{L_1}}(z)$

Step 2: Given $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$, $f_{\chi_{L_1}}(z)$, $f_{\chi_{L_1} \odot \chi_{L_2}}(z) = f_{\chi_{L_1}}(z)$?

Goal: Show that $f_{\chi_{L_1}}(z)$, $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ belong to a class $\mathbf{Q}[[z]]_h$ and that Equivalence for $\mathbf{Q}[[z]]_h$ is decidable.

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Main Tool: Holonomic Formal Series

Definition

A formal series $\phi \in \mathbf{Q}[[\Sigma]]$ is *holonomic* if and only if there exist some polynomials

$$p_{ij} \in \mathbf{Q}[\Sigma], \quad 1 \leq i \leq \sharp\Sigma, \quad 0 \leq j \leq d_i, \quad p_{id_i} \neq 0,$$

such that for $1 \leq i \leq \sharp\Sigma$

$$\sum_{j=0}^{d_i} p_{ij} \partial_{\sigma_i}^{j} \phi = 0.$$

Holonomic Formal Series

Properties

Theorem

$$\mathbf{Q}[[\Sigma]]_r \subset \mathbf{Q}[[\Sigma]]_a \subset \mathbf{Q}[[\Sigma]]_h.$$

Theorem (closure properties)

$\mathbf{Q}[[\Sigma]]_h$ is closed w.r.t. $+, \cdot, \odot, \Delta_{ij}$ and substitution (with algebraic series).

Fact

Equivalence for $\mathbf{Q}[[\Sigma]]_h$ is decidable.

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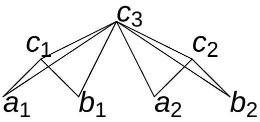
A Reduction for the Inclusion Problem

We reduce Inclusion for $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ with C given by



to Equivalence for $\mathbf{Q}[[z]]_h$ by showing that $f_{\chi_{L_1}}(z)$ and $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ are holonomic.

Outline

How to get $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$: a trivial case

Example (a null commutation relation)

Given

- $L_1, L_2 \in \mathrm{Rat}_{\mathrm{U}}(\{a,b\},\emptyset)$,
- χ_{L_1}, χ_{L_2} with linear representations $\langle \lambda_1, \mu_1, \gamma_1 \rangle$, $\langle \lambda_2, \mu_2, \gamma_2 \rangle$,

$$(\chi_{L_j}, x_1 \cdots x_n) = \lambda_j \mu_j(x_1) \cdots \mu_j(x_n) \gamma_j, \quad x_i \in \{a,b\}, \quad j = 1,2$$

we know that $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ is rational.

Fact

Let $w = x_1 \cdots x_n \in \{a,b\}^n$. Then

$$(\chi_{L_1} \odot \chi_{L_2}, w) = \lambda_1 \otimes \lambda_2 \cdot \mu_1(x_1) \otimes \mu_2(x_1) \cdots \mu_1(x_n) \otimes \mu_2(x_n) \cdot \gamma_1 \otimes \gamma_2$$

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How to get $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$: a trivial case

Let $\Lambda = \lambda_1 \otimes \lambda_2, \Gamma = \gamma_1 \otimes \gamma_2$. Then, $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$ is rational since

$$\begin{aligned} f_{\chi_{L_1} \odot \chi_{L_2}}(z) &= \sum_{w \in \{a,b\}^\star} (\chi_{L_1} \odot \chi_{L_2}, w) z^{|w|} = \\ &= \sum_{n \geq 0} z^n \sum_{w \in \{a,b\}^n} (\chi_{L_1} \odot \chi_{L_2}, w) = \\ &= \sum_{n \geq 0} z^n \Lambda \cdot (\mu_1(a) \otimes \mu_2(a) + \mu_1(b) \otimes \mu_2(b))^n \cdot \Gamma = \\ &= \Lambda (I - z(\mu_1(a) \otimes \mu_2(a) + \mu_1(b) \otimes \mu_2(b)))^{-1} \cdot \Gamma \end{aligned}$$

How to get $f_{\chi_{L_1}\odot\chi_{L_2}}(z)$: the case: $_a\overset{c}{\wedge}_b$

Given $\langle\lambda_1,\mu_1,\gamma_1\rangle, \langle\lambda_2,\mu_2,\gamma_2\rangle$ for L_1, L_2 in $\mathrm{Rat}_{\mathbb{U}}(\{a,b,c\}, {_a\overset{c}{\wedge}_b})$, note that

$$(\chi_{L_i},[\sigma_1\cdots\sigma_n c^k]_{\rho_C}) = \sum_{i_1+i_2\ldots+i_{n+1}=k} \lambda_i\mu_i(c^{i_1}\sigma_1 c^{i_2}\sigma_2\cdots c^{i_n}\sigma_n c^{i_{n+1}})\gamma_i$$

Step 1 For $\sigma\in\{a,b\}$ and $i=1,2$, define the matrices

$$A_\sigma^{(i)}(c) = \mu_i(\sigma)\cdot\sum_{k\geq 0} c^k\mu_i(c)^k = \mu_i(\sigma)\cdot(I-c\mu_i(c))^{-1}$$

and the row vectors

$$\Lambda_i(c) = \lambda_i\sum_{k\geq 0} c^k\mu_i(c)^k = \lambda_i\cdot(I-c\mu_i(c))^{-1}$$

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How to get $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$: the case: $\overset{c}{{}_a\wedge_b}$

Step 2 Define

$$\begin{aligned} A_\sigma(c_1, c_2) &= A_\sigma^{(1)}(c_1) \otimes A_\sigma^{(2)}(c_2), \\ \Upsilon(c_1, c_2) &= \Lambda_1(c_1) \otimes \Lambda_2(c_2), \\ \Gamma &= \gamma_1 \otimes \gamma_2 \end{aligned}$$

and observe that

$$\begin{aligned} (\chi_{L_1} \odot \chi_{L_2}, [\sigma_1 \cdots \sigma_n c^k]_{\rho_C}) &= [c_1^k c_2^k](\Upsilon(c_1, c_2) \cdot A_{\sigma_1}(c_1, c_2) \\ &\quad \cdots A_{\sigma_n}(c_1, c_2) \cdot \Gamma). \end{aligned}$$

How to get $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$: the case: $a \overset{c}{\wedge} b$

Step 3

$$
\begin{aligned}
f_{\chi_{L_1} \odot \chi_{L_2}}(z) &= \sum_{\substack{w \in \{a,b\}^\star \\ k \geq 0}} (\chi_{L_1} \odot \chi_{L_2}, [wc^k]_{\rho_C}) z^{|w|+k} = \\
&= \sum_{\substack{n \geq 0 \\ k \geq 0}} z^n z^k \sum_{w \in \{a,b\}^n} (\chi_{L_1} \odot \chi_{L_2}, [wc^k]_{\rho_C}) = \\
&= \sum_{k \geq 0} z^k [c_1^k c_2^k] \\
& \sum_{\substack{n \geq 0 \\ \sigma_1 \cdots \sigma_n \in \{a,b\}^n}} z^n \Upsilon(c_1, c_2) \cdot A_{\sigma_1}(c_1, c_2) \cdots A_{\sigma_n}(c_1, c_2) \cdot \Gamma
\end{aligned}
$$

How to get $f_{\chi_{L_1} \odot \chi_{L_2}}(z)$: the case: ${}_a\overset{c}{\wedge}_b$

$$
\begin{aligned}
&= \sum_{k \geq 0} z^k [c_1^k c_2^k] \Upsilon(c_1, c_2)(I - (A_a(c_1, c_2) + A_b(c_1, c_2))z)^{-1}\Gamma = \\
&= F(z, z)
\end{aligned}
$$

where

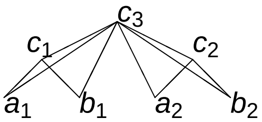
$$F(c_1, z) = \Delta_{c_1 c_2}(\Upsilon(c_1, c_2)(I - (A_a(c_1, c_2) + A_b(c_1, c_2))z)^{-1}\Gamma)$$

is algebraic (diagonal of a rational function).

Outline

A partition of $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$

Let T_7 be the relation



and partition $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$ in 4 classes

A partition of $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$

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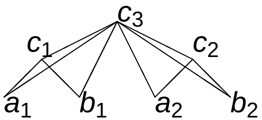
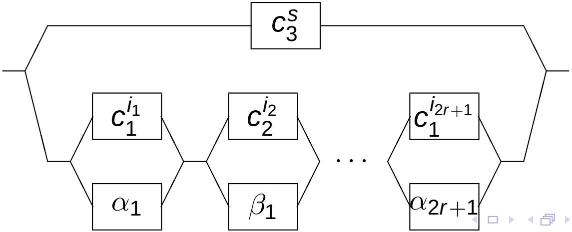


and partition $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$ in 4 classes

M_1:

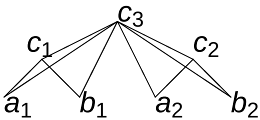


$\alpha_i \in \{a_1, b_1\}^\star$

$\beta_i \in \{a_2, b_2\}^\star$

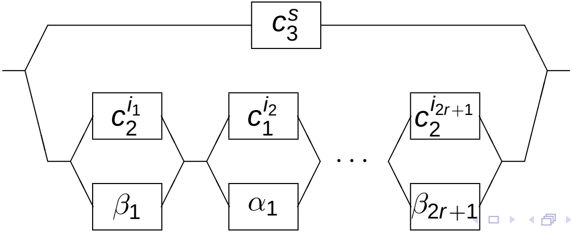
A partition of $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$

Let T_7 be the relation



and partition $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$ in 4 classes

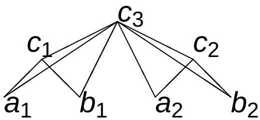
M_2:



$\alpha_i \in \{a_1, b_1\}^\star$
$\beta_i \in \{a_2, b_2\}^\star$

A partition of $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$

Let T_7 be the relation



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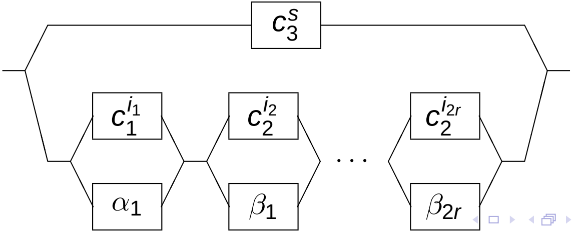
M_3:



$\alpha_i \in \{a_1, b_1\}^\star$

$\beta_i \in \{a_2, b_2\}^\star$

A partition of $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$

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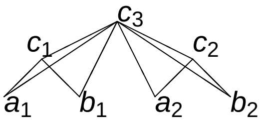
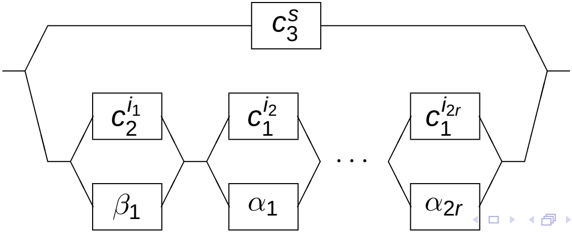


and partition $F(\{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}, T_7)$ in 4 classes

M_4:



$\alpha_i \in \{a_1, b_1\}^\star$

$\beta_i \in \{a_2, b_2\}^\star$

Fact

$$f_{\chi_{L_1}\odot\chi_{L_2}}(z) = f^{(1)}_{\chi_{L_1}\odot\chi_{L_2}}(z) + f^{(2)}_{\chi_{L_1}\odot\chi_{L_2}}(z) + f^{(3)}_{\chi_{L_1}\odot\chi_{L_2}}(z) + f^{(4)}_{\chi_{L_1}\odot\chi_{L_2}}(z)$$

where

$$f^{(i)}_{\chi_{L_1}\odot\chi_{L_2}}(z) = \sum_{t\in M_i} (\chi_{L_1} \odot \chi_{L_2}, t) z^{|t|}$$

Fact

$$f^{(i)}_{\chi_{L_1}\odot\chi_{L_2}}(z) \textit{ holonomic} \quad \Rightarrow \quad f_{\chi_{L_1}\odot\chi_{L_2}}(z) \textit{ holonomic}$$

(Closure properties of $\mathbf{Q}[[\Sigma]]_h$)

Fact

$$f_{\chi_{L_1} \odot \chi_{L_2}}(z) = f^{(1)}_{\chi_{L_1} \odot \chi_{L_2}}(z) + f^{(2)}_{\chi_{L_1} \odot \chi_{L_2}}(z) + f^{(3)}_{\chi_{L_1} \odot \chi_{L_2}}(z) + f^{(4)}_{\chi_{L_1} \odot \chi_{L_2}}(z)$$

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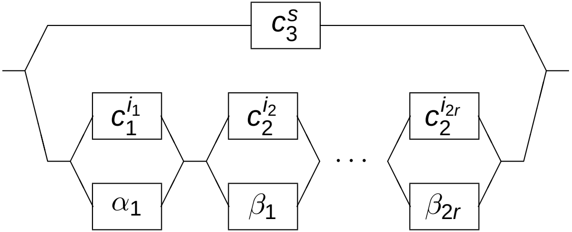
(Closure properties of $\mathbf{Q}[[\Sigma]]_h$)

How to get $f^{(3)}_{\chi_{L_1} \odot \chi_{L_2}}(z)$

Note that if $t \in M_3$

M_3:



$\alpha_i \in \{a_1, b_1\}^\star$

$\beta_i \in \{a_2, b_2\}^\star$

then $t = [w_1 w_2 \cdots w_{2r}]_{\rho_{T_7}}$ with

$$w_{2i} \in [\beta\delta y]_{\rho_{T_7}}, \quad \beta \in \{a_2, b_2\}^\star, \delta \in c_2^\star, y \in c_3^\star \quad \beta \neq \epsilon \vee \delta \neq \epsilon$$

$$w_{2i+1} \in [\alpha\gamma y]_{\rho_{T_7}}, \quad \alpha \in \{a_1, b_1\}^\star, \gamma \in c_1^\star, y \in c_3^\star \quad \alpha \neq \epsilon \vee \gamma \neq \epsilon$$

How to get $f^{(3)}_{\chi_{L_1}\odot\chi_{L_2}}(z)$: details

Step 1 Given $\langle\lambda_1,\mu_1,\gamma_1\rangle$, $\langle\lambda_2,\mu_2,\gamma_2\rangle$, let $\Lambda = \lambda_1 \otimes \lambda_2$, $\Gamma = \gamma_1 \otimes \gamma_2$. For $\sigma \in \{a_1, b_1, a_2, b_2\}$ and $i, j \in \{1, 2\}$ define the matrices

$$\begin{aligned}
C_{ij}(c_i, c_3) &= \sum_{k\geq 0}(c_i\mu_j(c_i) + c_3\mu_j(c_3))^k = \\
&= (I - (c_i\mu_j(c_i) + c_3\mu_j(c_3))^{-1}, \\
D_i(c_i, c_3, c_i', c_3') &= C_{i1}(c_i, c_3) \otimes C_{i2}(c_i', c_3'), \\
A_{ij\sigma}(c_i, c_3) &= \mu_j(\sigma)C_{ij}(c_i, c_3), \\
B_{i\sigma}(c_i, c_3, c_i', c_3') &= A_{i1\sigma}(c_i, c_3) \otimes A_{i2\sigma}(c_i', c_3'), \\
T_i(z, c_1, c_3, c_1', c_3') &= (I - z(B_{ia_i}(c_i, c_3, c_i', c_3') + \\
&\quad B_{ib_i}(c_i, c_3, c_i', c_3')))^{-1} - I,
\end{aligned}$$

How to get $f^{(3)}_{\chi_{L_1}\odot\chi_{L_2}}(z)$: details

Step 2 Define the matrix (algebraic entries)

$$\xi = \sum_{n\geq 0}(\Delta_{c_1,c_1'}T_1(\Delta_{c_2,c_2'}D_2+\Delta_{c_2,c_2'}T_2)+\Delta_{c_1,c_1'}D_1(\Delta_{c_2,c_2'}T_2+\Delta_{c_2,c_2'}D_2))^n$$

and note that $f^{(3)}_{\chi_{L_1}\odot\chi_{L_2}}(z) = F_3(z,z,z,z)$ where

$$F_3(c_1,c_2,c_3,z) = \Delta_{c_3,c_3'}(\Lambda\xi\Gamma)$$

is holonomic (diagonal of an algebraic series).

Decidability for Inclusion

Fact

$f^{(i)}_{\chi_{L_1}\odot\chi_{L_2}}(z)$ and $f_{\chi_{L_1}\odot\chi_{L_2}}(z)$ are holonomic.

Since Inclusion for $\mathrm{Rat}_{\mathrm{U}}(\{a_1, b_1, c_1, a_2, b_2, c_2, c_3\}, T_7)$ is reducible to Equivalence for $\mathbf{Q}[[\Sigma]]_h$ we have:

Theorem

Inclusion for $Rat_U(\{a_1, b_1, c_1, a_2, b_2, c_2, c_3\}, T_7)$ is decidable.

Conclusions and future works

- We can adapt the technique to work with $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ if C is described by a tree of height 2 (a and b commute iff a is in the subtree of b or vice versa).
- Extend the technique to $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ with C described by a tree of arbitrary height.

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- Extend the technique to $\mathrm{Rat}_{\mathrm{U}}(\Sigma, C)$ with C described by a tree of arbitrary height.