Unambiguous Morphic Images of Strings

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The task

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Map a finite string α over an infinite alphabet onto a finite word w over a binary alphabet such that w reflects the structure of α "optimally".

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Example: Let $\alpha = 1\ 2\ 3\ 4\ 1\ 4\ 3\ 2$. Let $\sigma(i) = \mathtt{ab}^i$.

$$\Rightarrow \sigma(\alpha) = \mathtt{ab}\ \mathtt{ab}^2\ \mathtt{ab}^3\ \mathtt{ab}^4\ \mathtt{ab}\ \mathtt{ab}^4\ \mathtt{ab}^3\ \mathtt{ab}^2$$

Ambiguity of words

Assumption: We are confronted with a variety of morphic images of a single (unknown) string α ($\rightarrow$ inductive inference of pattern languages, PCP).

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It does not prove the existence of the symbol "2" in α.

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Note: Ambiguity of morphic images can cause that, even in a weak sense, "decoding" is a non-computable problem [Reidenbach, 2002].

A simple combinatorial question on morphisms in free monoids

Motivation

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Results

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The Morphism

•

Conclusion

Question:

For which pattern α is there a morphism $\sigma: \mathbb{N}^+ \to \{\mathtt{a}, \mathtt{b}\}^*$ such that $\sigma(\alpha)$ is unambiguous, i.e. there is no morphism $\rho: \mathbb{N}^+ \to \{\mathtt{a}, \mathtt{b}\}^*$ with, for some variable i in α, $\rho(i) \neq \sigma(i)$ and $\rho(\alpha) = \sigma(\alpha)$?

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Remark: Question can easily be rephrased in terms of, e.g.,

- pattern languages and
- the Post Correspondence Problem, i.e. equality sets.

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Remark: Question can easily be rephrased in terms of, e.g.,

- pattern languages and
- the Post Correspondence Problem, i.e. equality sets.

Additionally, we provide an answer on finite fixed points of nontrivial morphisms...

Is this a trivial problem?

Examples suggest that this might be a nontrivial problem:

- Let $\alpha = 1\ 2\ 3\ 4\ 1\ 4\ 3\ 2$.

  $\Rightarrow$ We know that the image of the morphism $\sigma(i) = \mathbf{ab}^i$ is ambiguous.

  aabbabba is unambiguous, but the corresponding morphism is not injective.

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  By the way, **abbaaabb** is ambiguous...

- Let $\alpha = 1\,2$.

  ⇒ Evidently, there is no unambiguous morphic image:

  Let $\sigma(1) \neq \varepsilon$. Then, with $\rho(1) = \varepsilon$ and $\rho(2) = \sigma(1)\,\sigma(2)$, $\rho \neq \sigma$ and $\rho(\alpha) = \sigma(\alpha)$.

A first negative result and its consequences

Theorem:

There is no nonerasing morphism σ such that, for every pattern α, $\sigma(\alpha)$ is unambiguous.

A first negative result and its consequences

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$\sigma: \mathbb{N}^* \to \{\mathtt{a}, \mathtt{b}\}^+$

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Remarks:

Obviously, for each pattern $\alpha = i\,j$ (with $i, j \in \mathbb{N}$ and $i \neq j$) and for each nonerasing morphism σ, $\sigma(\alpha)$ is unambiguous.

We should generalise the structure of α...

A crucial (and natural) partition of $\mathbb{N}^+$

Definition:

We call any $\alpha \in \mathbb{N}^+$ prolix iff there exists a decomposition

$$\alpha = \beta_0 \gamma_1 \beta_1 \gamma_2 \beta_2 \dots \beta_{n-1} \gamma_n \beta_n$$

with $n \geq 1$, $\beta_k \in \mathbb{N}^*$ and $\gamma_k \in \mathbb{N}^+$, $k \leq n$, such that

1. for every k, $1 \leq k \leq n$, $|\gamma_k| \geq 2$,
2. for every k, $1 \leq k \leq n$, and for every k', $0 \leq k' \leq n$, var $(\gamma_k) \cap$ var $(\beta_{k'}) = \varnothing$ and
3. for every k, $1 \leq k \leq n$, there exists an $i \in$ var (γ_k) such that $|\gamma_k|_i = 1$ and, for every k', $1 \leq k' \leq n$, if $i \in$ var $(\gamma_{k'})$ then $\gamma_k = \gamma_{k'}$.

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Examples: 1 2 1 2 3 1 1 4 4 3 1 1 is prolix. 1 2 1 2 2 is succinct.

Why is this a natural partition?

A succinct pattern is a shortest generator of a terminal-free E-pattern language:

Theorem [Reidenbach, 2002]:

A pattern α is succinct if and only if, for every pattern α' with $L(\alpha') = L(\alpha)$, $|\alpha'| \geq |\alpha|$.

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The set of prolix patterns exactly corresponds to the set of finite fixed points of nontrivial morphisms [Head, 1981]:

Theorem [Head, 1981]:

> A pattern α is prolix if and only if there exists a morphism $\varphi: \mathbb{N}^* \rightarrow \mathbb{N}^*$ such that $\varphi \neq id$ and $\varphi(\alpha) = \alpha$.

Unambiguous words for prolix patterns? No...

Theorem:

For any prolix pattern α and for any nonerasing morphism σ, $\sigma(\alpha)$ is ambiguous.

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Proof:

α is prolix

$\Rightarrow$ there is a φ with $\varphi(\alpha) = \alpha$ and $\varphi(i) = \varepsilon$ for some i in var(α)

$\Rightarrow$ for every σ define $\rho = \sigma \circ \varphi$

$\Rightarrow$ $\rho(\alpha) = \sigma(\alpha)$, but ρ is not nonerasing

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...and yes:
If we omit the claim that σ must be nonerasing then some prolix patterns have an unambiguous image, some have not.

A first approach to succinct patterns

Proposition:

There is no nonerasing morphism σ such that, for every succinct pattern α, $\sigma(\alpha)$ is unambiguous.

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Proof idea:

Particular types of (sub-)patterns in α are required such as

$$\alpha = \dots j \;\; k \;\; j \;\; k' \;\; j' \;\; k \;\; j' \;\; k' \dots,$$

with no other occurrences of these variables.

For every nonerasing morphism σ we can find a pattern α such that $\sigma(\alpha)$ is not unambiguous.

The main result

Theorem:

For every succinct pattern α, there is an injective morphism σ_α such that $\sigma_\alpha(\alpha)$ is unambiguous.

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This leads to a third characterisation of succinctness:

Corollary:

A pattern α is succinct if and only if there exists an injective morphism σ such that $\sigma(\alpha)$ is unambiguous.

The morphism – some ideas [1 of 7]

Recall the above structure: Let $\alpha = 1\ 2\ 1\ 3\ 4\ 2\ 4\ 3$.

`b a b b a a a b a a`

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The morphism – some ideas [1 of 7]

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First observation:

In the given example, ambiguity is caused by the fact that the morphic images of "1" and "4" end with the same letter, namely `a`.

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Second observation:

For every occurrence of "1" and "4", the right neighbour reads either "2" or "3", and the latter variables have no other left neighbours in α.

The morphism – some ideas [2 of 7]

A first idea...

For every variable i in a pattern α, identify the set L_i (resp. R_i) of all of its left (resp. right) neighbours. Choose σ_α such that each L_i (resp. R_i) contains variables that have morphic images with different last (resp. first) letters.

In other words: Choose σ_α such that each L_i (resp. R_i) is (morphically) heterogeneous.

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...that does not work:

Example: Let $\alpha = 1\ 2\ 3\ 2\ 1\ 3\ 1$.

$\rightarrow$ $L_1 = \{2, 3\}$, $L_2 = \{1, 3\}$, $L_3 = \{1, 2\}$

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$\Rightarrow$ $\sigma_\alpha(2) = \ldots$ **a**
$\sigma_\alpha(3) = \ldots$ **b**

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$\Rightarrow$ $\sigma_\alpha(2) = \ldots \mathtt{a}$, $\sigma_\alpha(3) = \ldots \mathtt{b}$; $\sigma_\alpha(1) = \ldots \mathtt{a}$, $\sigma_\alpha(3) = \ldots \mathtt{b}$

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$\sigma_\alpha(2) = \dots \mathtt{a}$, $\sigma_\alpha(3) = \dots \mathtt{b}$; $\sigma_\alpha(1) = \dots \mathtt{a}$, $\sigma_\alpha(3) = \dots \mathtt{b}$; $\sigma_\alpha(1) = \dots \mathtt{a}$, $\sigma_\alpha(2) = \dots \mathtt{a}$

The morphism – some ideas [3 of 7]

Question:

Can we weaken the claim that each L_i (resp. R_i) must be morphically heterogeneous?

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Let $\alpha = 1\ \underline{2}\ 3\ \underline{2}\ 4\ \underline{5}\ 3\ \underline{5}\ 1\ \underline{6}\ 4\ \underline{6}$.

$\rightarrow \quad L_2 = \{1, 3\}, \quad L_5 = \{4, 3\}, \quad L_6 = \{1, 4\}$

... **a** $\sigma_\alpha(2)$ ... **b** $\sigma_\alpha(2)$ ... **a** $\sigma_\alpha(5)$ ... **b** $\sigma_\alpha(5)$ ... **a** $\sigma_\alpha(6)$ ... **a** $\sigma_\alpha(6)$

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$\ldots\ \mathtt{a}\ \sigma_\alpha(2) \ldots \mathtt{b}\ \sigma_\alpha(2) \ldots \mathtt{a}\ \sigma_\alpha(5) \ldots \mathtt{b}\ \sigma_\alpha(5) \ldots \mathtt{a}\ \sigma_\alpha(6) \ldots \mathtt{a}\ \sigma_\alpha(6)$

$\rho\ (6)$

$\Rightarrow \qquad \rho\ (6) = \mathtt{a}\ \sigma_\alpha(6)$

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Let $\alpha = 1\ \underline{2}\ 3\ \underline{2}\ 4\ \underline{5}\ 3\ \underline{5}\ 1\ \underline{6}\ 4\ \underline{6}$.

$\rightarrow$ $L_2 = \{1, 3\}$, $L_5 = \{4, 3\}$, $L_6 = \{1, 4\}$

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but then $\rho(5) = \mathtt{b}\ \sigma_\alpha(5)$. Thus, there is no morphism ρ.

The morphism – some ideas [4 of 7]

Observation:

Whenever, for some $j, k \in$ var (α), $L_j \cap L_k \neq \varnothing$ then the heterogeneity of L_j "protects" the images of the variables in L_k and vice versa.

The morphism – some ideas [4 of 7]

Observation:

Whenever, for some $j, k \in$ var (α), $L_j \cap L_k \neq \varnothing$ then the heterogeneity of L_j "protects" the images of the variables in L_k and vice versa.

In fact, this property even is "transitive", i.e. if, for some $j, k \in$ var (α), there exist arbitrary $L_{k,1}$, $L_{k,2}$, ..., $L_{k,p}$ with

- $L_{k,i} \cap L_{k,i+1} \neq \varnothing$ and
- $L_{k,1} = L_j$ and $L_{k,p} = L_k$

then the heterogeneity of L_j makes an alternative morphic image for k impossible.

The morphism – some ideas [5 of 7]

Idea:

We introduce an equivalence relation $\sim_L$ on var (α):

For each pair $j, j' \in$ var (α), $j \sim_L j'$ if and only if there exist $L_{k,1}, L_{k,2}, \ldots, L_{k,p}$ with

- $L_{k,i} \cap L_{k,i+1} \neq \emptyset$ and
- $j \in L_{k,1}$ and $j' \in L_{k,p}$.

We name the resulting equivalence classes $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_m$. Thus, $\mathscr{L}_1 \cup \mathscr{L}_2 \cup \ldots \cup \mathscr{L}_m =$ var (α), and the $\mathscr{L}_i$ are pairwise disjoint.

Remark:

Analogously, we introduce a relation $\sim_R$.

The morphism – some ideas [6 of 7]

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Example:

Let $\alpha = 1\ 2\ 3\ 1\ 2\ 3\ 1\ 3\ 4\ 5\ 6\ 4\ 6\ 5$.

$\rightarrow$ $L_1 = \{3\}$, $L_2 = \{1\}$, $L_3 = \{1, 2\}$,
$L_4 = \{3, 6\}$, $L_5 = \{4, 6\}$, $L_6 = \{4, 5\}$

$\rightarrow$ $\mathscr{L}_1 = \{1, 2\}$, $\mathscr{L}_2 = \{3, 4, 5, 6\}$

The morphism – some ideas [6 of 7]

Example:

Let $\alpha = 1\ 2\ 3\ 1\ 2\ 3\ 1\ 3\ 4\ 5\ 6\ 4\ 6\ 5$.

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$\rightarrow$ $\mathscr{L}_1 = \{1, 2\}$, $\mathscr{L}_2 = \{3, 4, 5, 6\}$

Conclusion:

For our needs, it is sufficient that each $\mathscr{L}_i$ (resp. $\mathscr{R}_i$) is morphically heterogeneous.

As these sets are pairwise disjoint, there are no conflicting assignments when composing σ_α.

The morphism – some ideas [7 of 7]

Motivation
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Conclusion:

For every succinct pattern α, the following morphism σ_α generates an unambiguous word:

$$\sigma_\alpha(k) = \begin{cases} \mathtt{a}\,\mathtt{b}^{3k}\,\mathtt{a}\;\mathtt{a}\,\mathtt{b}^{3k+1}\,\mathtt{a}\;\mathtt{a}\,\mathtt{b}^{3k+2}\,\mathtt{a}, & \nexists\, i: k = \min \mathscr{L}_i \wedge \nexists\, j: k = \min \mathscr{R}_j, \\ \mathtt{b}\,\mathtt{a}^{3k}\,\mathtt{b}\;\mathtt{a}\,\mathtt{b}^{3k+1}\,\mathtt{a}\;\mathtt{a}\,\mathtt{b}^{3k+2}\,\mathtt{a}, & \nexists\, i: k = \min \mathscr{L}_i \wedge \exists\, j: k = \min \mathscr{R}_j, \\ \mathtt{a}\,\mathtt{b}^{3k}\,\mathtt{a}\;\mathtt{a}\,\mathtt{b}^{3k+1}\,\mathtt{a}\;\mathtt{b}\,\mathtt{a}^{3k+2}\,\mathtt{b}, & \exists\, i: k = \min \mathscr{L}_i \wedge \nexists\, j: k = \min \mathscr{R}_j, \\ \mathtt{b}\,\mathtt{a}^{3k}\,\mathtt{b}\;\mathtt{a}\,\mathtt{b}^{3k+1}\,\mathtt{a}\;\mathtt{b}\,\mathtt{a}^{3k+2}\,\mathtt{b}, & \exists\, i: k = \min \mathscr{L}_i \wedge \exists\, j: k = \min \mathscr{R}_j, \end{cases}$$

$k \in \mathrm{var}(\alpha)$.

Conclusion

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1. There is no nonerasing morphism which, for every pattern, generates an unambiguous word.

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Conclusion

1. There is no nonerasing morphism which, for every pattern, generates an unambiguous word.
2. The image of any prolix pattern under any nonerasing morphism is not unambiguous.
3. There is no nonerasing morphism which, for every succinct pattern, generates an unambiguous word.
4. For every succinct pattern there is an injective morphism generating an unambiguous word.