Palindromes in Sturmian Words

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Outline

- Preliminaries
  - Notation
  - Sturmian words
  - Central words
- Counting Sturmian palindromes
- Periods of Sturmian palindromes
- Two structural results

Notation

For a word over w over some alphabet A ($w \in A^*$), we denote by π_w the *minimal period* of w.
Thus if $w = a_1 a_2 \cdots a_n$ with $a_1, \ldots, a_n \in A$ then:

$$i \equiv j \pmod{\pi_w} \Rightarrow a_i = a_j$$

for all $i, j \leq n$.
The *reversal* of w is $\tilde{w} = a_n \cdots a_1$.

Sturmian Words

An infinite word x is *Sturmian* if it has $n + 1$ factors of any length n:

$$\mathrm{card}(\mathrm{Fact}(x) \cap \quad {}^{n}) = n + 1 \ .$$

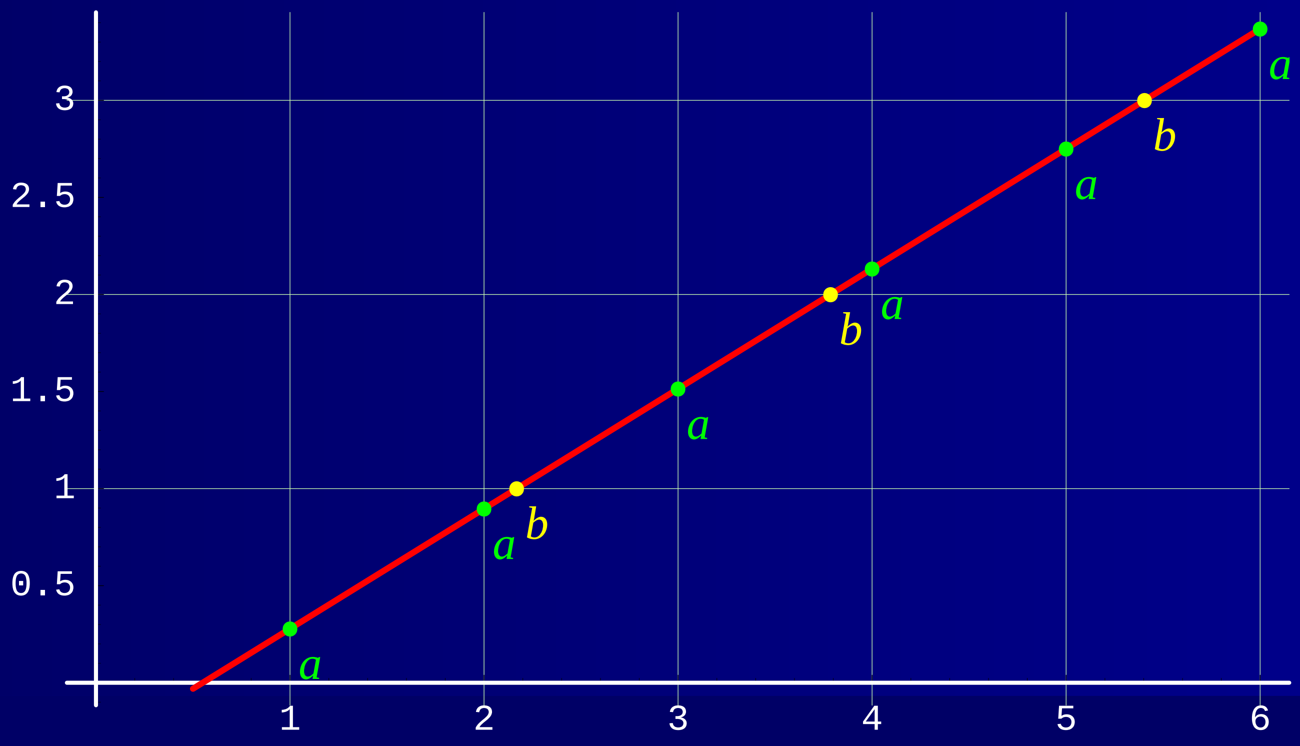
In particular, Sturmian words are over a 2-letter alphabet. We assume $\quad = \{a, b\}$.

Sturmian words have many different characterizations.

A geometric view

Sturmian words are exactly the *cutting sequences* of half lines having positive irrational slope:

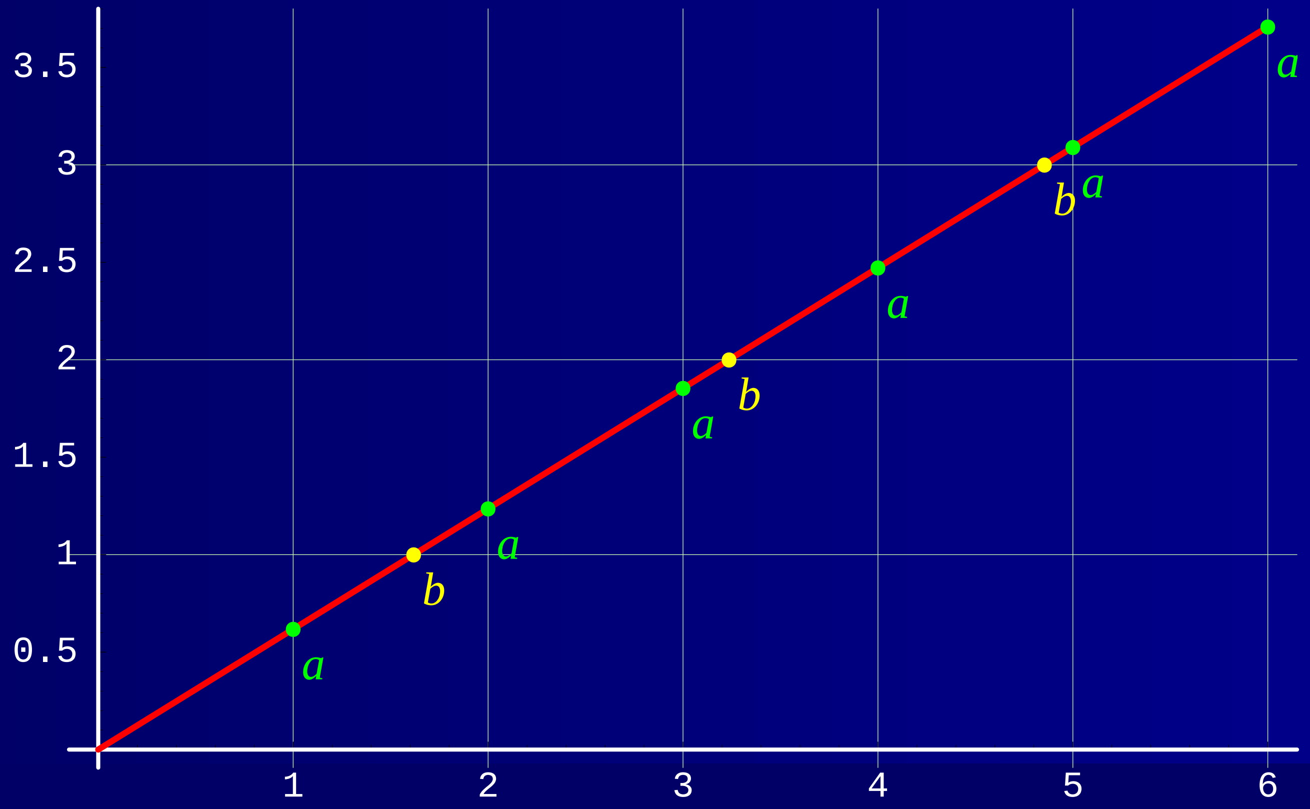
$x = aababaaba\cdots$



A geometric view

Words represented by half lines starting at the origin are called *standard*.

$f = abaababaa\cdots$



More definitions

We set $PAL = \{w \in \quad {}^* \mid w = \tilde{w}\}$, and denote by St the set of factors of all Sturmian words. The elements of St are called (finite) Sturmian words, or sometimes *balanced* words.

Our contribution is focused on the set

$$St \cap PAL$$

of *Sturmian palindromes*.

Central words

The set of palindromic prefixes of standard words is denoted by PER, and its elements are called *central words*. They are characterized by the following properties:

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- $w \in PER \Leftrightarrow w \in a^* \cup b^*$ or $w = w_1abw_2 = w_2baw_1$ for some $w_1, w_2 \in \;\;{}^*$.

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Trivially, $PER \subseteq St \cap PAL$, but

$$abba \in (St \cap PAL) \setminus PER \neq \emptyset \ .$$

Central words

Moreover, $\pi_w = \min\{p, q\}$. When $\pi_w > 1$, one has also $\{p, q\} = \{|w_1| + 2, |w_2| + 2\}$, and $w_1, w_2 \in PER$.

The number of central words for each length is already known. Our first goal is to compute the function g defined by

$$g(n) = \mathrm{card}(St \cap PAL \cap \quad ^{n}) \quad (n \geq 0) \;,$$

that is, to count Sturmian palindromes.

Counting Sturmian palindromes

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- The number of central words of length n is:

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It is easy to prove that if $w \in (St \cap PAL) \setminus PER$, then either awa or bwb is Sturmian, not both.

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Hence,

$$g(n+2) = g(n) + \phi(n+2)\ .$$

The enumeration formula

In conclusion, from the conditions

$$\begin{cases} g(0) &= 1 \\ g(1) &= 2 \\ g(n+2) &= g(n) + \phi(n+2) \end{cases}$$

one can derive by recurrence the following equations.

The enumeration formula

For any integer $n \geq 0$, one has:

$$g(2n) = 1 + \sum_{k=1}^{n} \phi(2k)$$

and

$$g(2n+1) = 1 + \sum_{k=0}^{n} \phi(2k+1) \ .$$

Example

As an example, these are the $g(7) = 14$ Sturmian palindromes of length 7:

$aaaaaaa$	$bbbbbbb$
$aaabaaa$	$bbbabbb$
$aababaa$	$bbababb$
$abaaaba$	$babbbab$
$abababa$	$bababab$
$abbabba$	$baabaab$
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$abababa$	$bababab$
$abbabba$	$baabaab$
$abbbbba$	$baaaaab$

with the $\phi(7+2) = 6$ central words in yellow.

Median factors

Every finite Sturmian word is a factor of some central word:

$$St = \mathrm{Fact}(PER) \; .$$

Similarly, every Sturmian palindrome is a *median* factor of a central word:

$$w \in St \cap PAL \Leftrightarrow \exists\, \mu \in \quad {}^* : \; \mu w \tilde{\mu} \in PER \; .$$

Relating the periods

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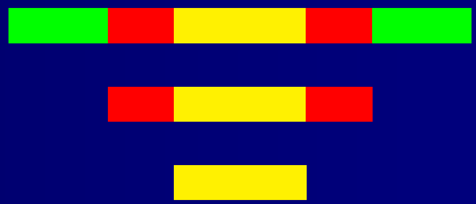
Theorem. With the above definitions, one has $\pi_w = \pi_u$, and either

$$\pi_w = \pi_v \text{ or } \pi_w = |v| + 2 - \pi_v \quad .$$

Proof of $\pi_w = \pi_u \in \{\pi_v, |v| + 2 - \pi_v\}$



$u \in PER$

$w \in St \cap PAL$

$v \in PER$

If $\pi_v = 1$, then one may assume $v = a^n$. If $w \neq v$, then $n > 1$ and

$$v_1 := ba^n b$$

is a median factor of w.

Proof of $\pi_w = \pi_u \in \{\pi_v, |v| + 2 - \pi_v\}$

The only median extensions in St of $ba^n b$ are:

$$\begin{aligned} v_2 &= aba^nba \ , \\ v_3 &= a^2ba^nba^2 \ , \\ &\vdots \\ v_n &= a^{n-1}ba^nba^{n-1} \in PER \ . \end{aligned}$$

Since $v_i \notin PER$ for $i < n$, we get $u = v_n$ and $w = v_j$ for some $j < n$, so that

$$\pi_w = \pi_u = n + 1 = |v| + 2 - \pi_v \ .$$

Proof of $\pi_w = \pi_u \in \{\pi_v, |v| + 2 - \pi_v\}$

If $\pi_v > 1$, we may assume

$$v = w_1 abw_2 = w_2 baw_1 \ ,$$

with $\pi_v = |w_1| + 2$.
If $w \neq v$, then one between $v_1 := bvb$ and $v_1' := ava$ is a median factor of w.

In the first case, suppose

$$w_1 = p_1 p_2 \cdots p_k = p_k \cdots p_1 \ ,$$

with $p_i \in$ for $1 \leq i \leq k$.

Proof of $\pi_w = \pi_u \in \{\pi_v, |v| + 2 - \pi_v\}$

Define:

$$
\begin{aligned}
v_2 &= av_1a = ab(w_1abw_2)ba \quad , \\
v_3 &= p_kv_2p_k = p_kab(w_1abw_2)bap_k \quad , \\
&\vdots \\
v_{k+1} &= p_2 \cdots p_kabw_1abw_2bap_k \cdots p_2 \quad , \\
v_{k+2} &= (w_1)ab(w_1abw_2baw_1) = \\
&= (w_1abw_2baw_1)ba(w_1) \in PER \quad ,
\end{aligned}
$$

so that $w, u \in \{v_1, \ldots, v_{k+2}\}$ and

$$\pi_w = \pi_u = |w_1ab| = \pi_v.$$

Conclusion of the proof

In the second case ($v_1' = ava \in \mathtt{Fact}(w)$), we interchange the roles of w_1 and w_2. Assuming $w_2 = q_1 \cdots q_h$, and defining

$$\begin{aligned}
v_2' &= bv_1'b = baw_2baw_1ab \\
&\vdots \\
v_{h+1}' &= q_2 \cdots q_k baw_2baw_1abq_k \cdots q_2 \\
v_{h+2}' &= (w_2)ba(w_2baw_1abw_2) \in PER\ .
\end{aligned}$$

one eventually obtains

$\pi_w = \pi_u = |w_2ba| = |v| + 2 - \pi_v$. $\square$

Special factors

Let $f, w \in \quad {}^*$. We recall that f is called a *right special* factor of w if fa and fb are both factors of w.

We denote by R_w the minimal integer such that w has no right special factor of length R_w. For example, $R_{aab} = 2$. The following remarkable relation holds for any $w \in \quad {}^*$:

$$\pi_w \geq R_w + 1 \quad .$$

Two structural results

The previous theorem on periodicity leads to the following characterizations:

- A palindrome $w \in \quad {}^*$ with $\pi_w > 1$ is central if and only if its prefix of length $\pi_w - 2$ is a right special factor.

Two structural results

The previous theorem on periodicity leads to the following characterizations:

- A palindrome $w \in \quad {}^*$ with $\pi_w > 1$ is central if and only if its prefix of length $\pi_w - 2$ is a right special factor.
- A palindrome $w \in \quad {}^*$ is Sturmian if and only if $\pi_w = R_w + 1$.

Proof of $\pi_w = R_w + 1$

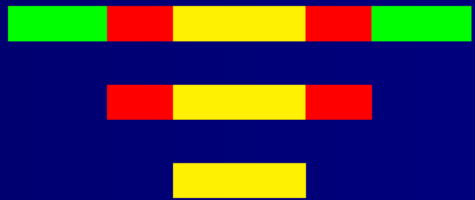
Trivial if $\pi_w = 1$. Assume $\pi_w > 1$. The condition $\pi_w = R_w + 1$ is equivalent to the inequality

$$\pi_w \leq R_w + 1$$

which expresses the existence of a right special factor s in w of length $\pi_w - 2$.

If $w \in PER$, then it has such a factor. Otherwise, define u and v as before.

Proof of $\pi_w = R_w + 1$



If $v = a^n$, then w has the factor ba^nb, so that a^{n-1} is a right special factor of w, of length $n - 1 = |v| - \pi_v = \pi_w - 2$.

Now let $v = w_1abw_2 = w_2baw_1$, $\pi_v = |w_1ab|$.

Proof of $\pi_w = R_w + 1$

If $\pi_w = \pi_v$, then w_1 is right special in $v = w_1abw_2 = w_2baw_1$, and then in w. One has $|w_1| = \pi_v - 2 = \pi_w - 2$.

If $\pi_w = |v| + 2 - \pi_v$, then w_2 is a right special factor of $ava = aw_1abw_2a = aw_2baw_1a$, which is a factor of w. In addition, $|w_2| = |v| - \pi_v = \pi_w - 2$.$\square$

An example

Let $w = aababaa \in St \cap PAL$. Then

$$\begin{aligned} u &= abwba = abaababaaba \ , \\ v &= ababa \ , \end{aligned}$$

so that $\pi_w = \pi_u = 5 = 5 + 2 - 2 = |v| + 2 - \pi_v$. The word aba is a right special factor of w of maximal length; hence, $R_w = 4 = \pi_w - 1$.

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