Bidimensional Sturmian Sequences and Substitutions

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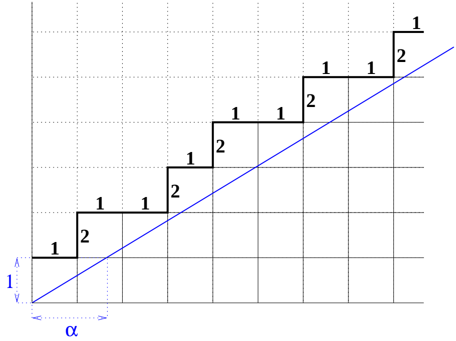
DLT'05

1. Prelude: Sturmian words and substitutions
2. Generalized substitutions
   - The linear map $\Theta(\sigma)$
   - The dual map $\Theta^*(\sigma)$
3. Bidimensional Sturmian sequences
   - Stepped planes and associated sequences
   - The action of Θ^*
4. Algebraic characterization
   - Bidimensional continued fractions
   - The case of periodic expansions

$\alpha \in \mathbb{R}\backslash\mathbb{Q} \rightsquigarrow$ Sturmian word u_α:



Here: $\alpha = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{\ddots}} \rightsquigarrow u_\alpha = 12112121121121\cdots$

Substitution: morphism σ of $\mathcal{A}^*$ s.t. $|\sigma^n(i)| \to \infty$ for $i \in \mathcal{A}$.

$\sigma : 1 \mapsto 12, 2 \mapsto 1$:

$$1 \to 12 \to 121 \to 12112 \to 12112121 \to \dots$$

σ extended to $\mathcal{A}^\omega \rightsquigarrow$ fixed-point: $u \in \mathcal{A}^\omega \mid u = \sigma(u)$.

$$\lim_{n\to\infty} \sigma^n(1) = 12112121121121\cdots = u_\alpha = \sigma(u_\alpha).$$

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Theorem (Algebraic Characterization I)

The Sturmian word u_α is a fixed-point if and only if α has a periodic continued fraction expansion.

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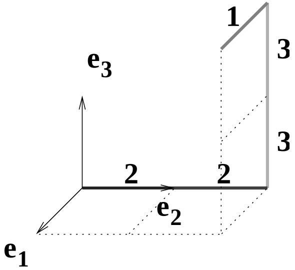
$\mathcal{A} = \{1, 2, 3\}$ and $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ canonical basis of $\mathbb{R}^3$.

$u \in \mathcal{A}^* \rightsquigarrow$ broken line of segments $[\vec{x}, \vec{x} + \vec{e}_i] = (\vec{x}, i)$, $\vec{x} \in \mathbb{N}^3$:

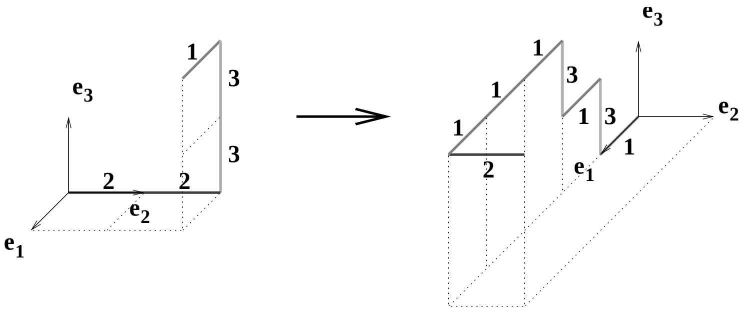


σ on $\mathcal{A} \rightsquigarrow$ linear map $\Theta(\sigma)$ on segments:

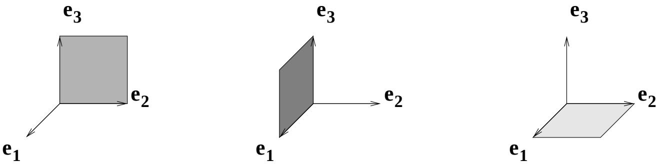
$$\Theta(\sigma) \; : \; (\vec{x}, i) \mapsto M_\sigma \vec{x} + \sum_{p|\sigma(i)=p\cdot j\cdot s} (\vec{f}(p), j),$$

where $(M_\sigma)_{i,j} = |\sigma(j)|_i$ and $\vec{f}(u) = {}^t(|u|_1, |u|_2, |u|_3)$.

$$\sigma : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 13 \\ 3 \mapsto 1 \end{cases}, \quad M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma(22331) = 13131112.$$



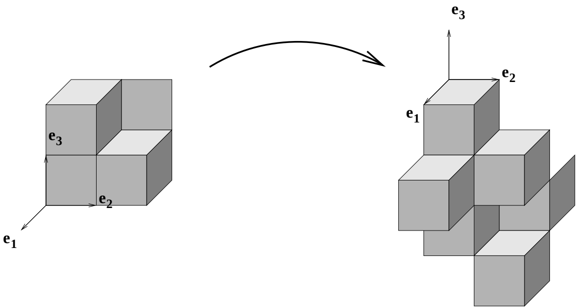
Segment $(\vec{x}, i) \rightsquigarrow$ dual face $(\vec{x}, i^*)$:



If $\det(M_\sigma) = \pm 1$: linear map $\Theta(\sigma) \rightsquigarrow$ dual map $\Theta^*(\sigma)$:

$$\Theta^*(\sigma)(\vec{x}, i^*) = M_\sigma^{-1}\vec{x} + \sum_{j \in \mathcal{A}} \sum_{s | \sigma(j) = p \cdot i \cdot s} (\vec{f}(s), j^*).$$

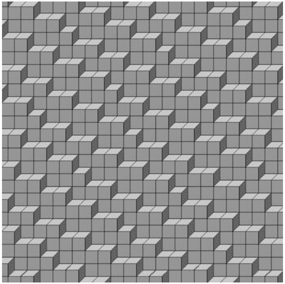
$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$:



$\alpha, \beta \in [0,1)^2$: $\mathcal{P}_{\alpha,\beta} = \{\vec{x} \in \mathbb{R}^3 \mid \langle \vec{x}, {}^t(1,\alpha,\beta)\rangle = 0\}$.

Definition (Stepped plane)

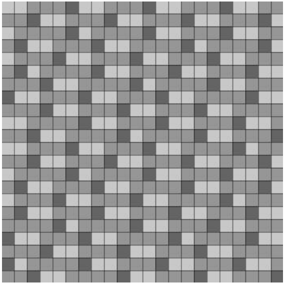
$$\mathcal{S}_{\alpha,\beta} = \{(\vec{x}, i^*) \mid 0 \leq \langle \vec{x}, {}^t(1,\alpha,\beta)\rangle < \langle \vec{e}_i, {}^t(1,\alpha,\beta)\rangle\}.$$



Theorem

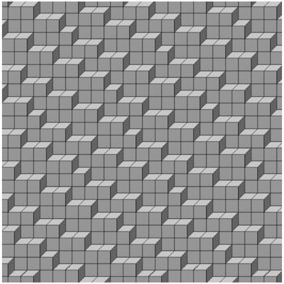
One can bijectively map the faces of the stepped plane $\mathcal{S}_{\alpha,\beta}$ to the letters of a bidimensional sequence $\mathcal{U}_{\alpha,\beta}$ over $\mathcal{A} = \{1, 2, 3\}$.

1, α and β linearly independent over $\mathbb{Q} \Rightarrow \mathcal{U}_{\alpha,\beta}$ *Sturmian*.

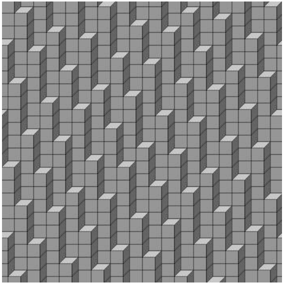


Theorem (Action of Θ^*)

$$^t(1,\alpha',\beta') \propto {}^tM_\sigma {}^t(1,\alpha,\beta) \Rightarrow \Theta^*(\sigma)(\mathcal{S}_{\alpha,\beta}) = \mathcal{S}_{\alpha',\beta'}.$$

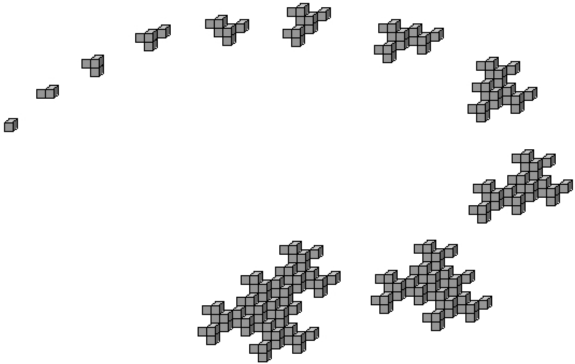


$\longmapsto$



$$M_\sigma^{-1}\mathcal{P}_{\alpha,\beta} = \mathcal{P}_{\alpha',\beta'} \rightsquigarrow \Theta^*(\sigma) \text{ “discretization” of } M_\sigma^{-1}.$$

$(\alpha, \beta) = (\alpha', \beta') \rightsquigarrow$ growing patches of $\mathcal{S}_{\alpha,\beta}$:



$\rightsquigarrow \Theta^*(\sigma)$ bidimensional substitution.

Modified Jacobi-Perron:

$$\begin{aligned}(\alpha,\beta) &= [(a_1,\varepsilon_1),\ldots,(a_k,\varepsilon_k),[(\alpha_k,\beta_k)]] \\ &= [(a_1,\varepsilon_1),(a_2,\varepsilon_2),\ldots]\end{aligned}$$

where $a_i \in \mathbb{N}$ and $\varepsilon_i \in \{0,1\}$.

Matrix viewpoint:

$$^t(1,\alpha_{k-1},\beta_{k-1}) = \underbrace{\eta_k}_{\in\mathbb{R}} {}^tM_{\sigma_{(a_k,\varepsilon_k)}} {}^t(1,\alpha_k,\beta_k),$$

where $\sigma_{(a_k,\varepsilon_k)}$ substitution on $\{1,2,3\}$ s.t. $\det(M_{\sigma_{(a_k,\varepsilon_k)}}) = 1$.

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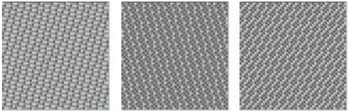
By theorem (Action of Θ^*):

$$\Theta^*(\sigma_{(a_k,\varepsilon_k)})(\mathcal{S}_{\alpha_k,\beta_k}) = \mathcal{S}_{\alpha_{k-1},\beta_{k-1}}.$$

Then, $\Theta^*(\sigma\sigma') = \Theta^*(\sigma')\Theta^*(\sigma)$ yields:

$$\Theta^*(\sigma_{(a_k,\varepsilon_k)}\ldots\sigma_{(a_1,\varepsilon_1)})(\mathcal{S}_{\alpha_k,\beta_k}) = \mathcal{S}_{\alpha,\beta}.$$



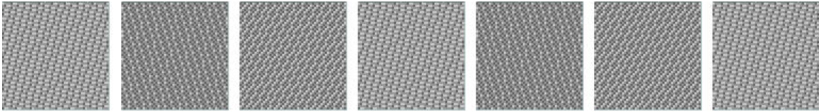
...

$(\alpha_p, \beta_p) = (\alpha, \beta) \rightsquigarrow$ periodic expansion:

$$(\alpha, \beta) = [(a_1, \varepsilon_1), \ldots, (a_p, \varepsilon_p), [(\alpha, \beta)]].$$

Then, $\mathcal{S}_{\alpha,\beta}$ fixed-point:

$$\Theta^*(\sigma_{(a_p,\varepsilon_p)} \cdots \sigma_{(a_1,\varepsilon_1)})(\underbrace{\mathcal{S}_{\alpha_p,\beta_p}}_{\mathcal{S}_{\alpha,\beta}}) = \mathcal{S}_{\alpha,\beta}.$$



$$(\alpha, \beta) = [(1,1), (1,1), (1,0), (1,1), (1,1), (1,0), (1,1), \ldots]$$

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What about the "only if" part?

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