STURMIAN WORDS: DYNAMICAL SYSTEMS AND DERIVATED WORDS

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Characteristic sturmian words

$\alpha \in]0,1[\setminus \mathbb{Q}$, with continued fraction $[0, 1 + a_1, a_2, \ldots]$

Define $(t_n)_n$

$$t_0 = 0,\ t_1 = 0^{a_1}1,\ t_n = t_{n-1}^{a_n} t_{n-2}\ (n \geqslant 2)$$

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- The Fibonacci word is a characteristic Sturmian word.

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- S_α is also the topological closure of

$$R_\alpha = \{\sigma^i(c_\alpha) \mid i \geqslant 0\}$$

where σ is the shift, and the closure of R_α is

$$\{x \in \{0,1\}^\omega \mid \mathrm{Pref}(x) \subseteq \mathrm{Pref}(R_\alpha)\}.$$

EXAMPLE

- c_α the characteristic Sturmian word of slope $\alpha = [0, \overline{3, 2}]$

$$c_\alpha = 0\,0\,1\,0\,0\,1\,0\,0\,0\,1\,0\,0\,1\,0\,0\,0\,1\,0\,0\,1\,0\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,0\,1 \cdots$$

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- The set R_α is formed by all suffixes of c_α
- $1c_\alpha,\ 0c_\alpha \in S_\alpha \backslash R_\alpha$

Return words

$x = x_0 x_1 \cdots$ a Sturmian word; w a factor of x

An integer i is an *occurrence* of w in x if

$$x_i x_{i+1} \cdots x_{i+|w|-1} = w.$$

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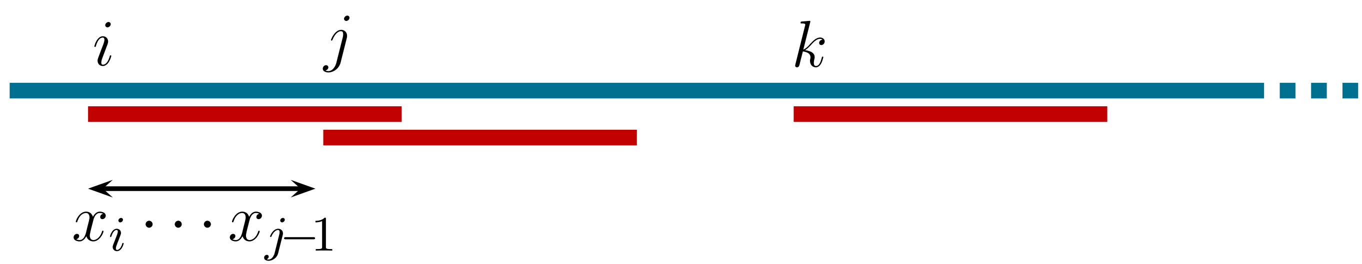

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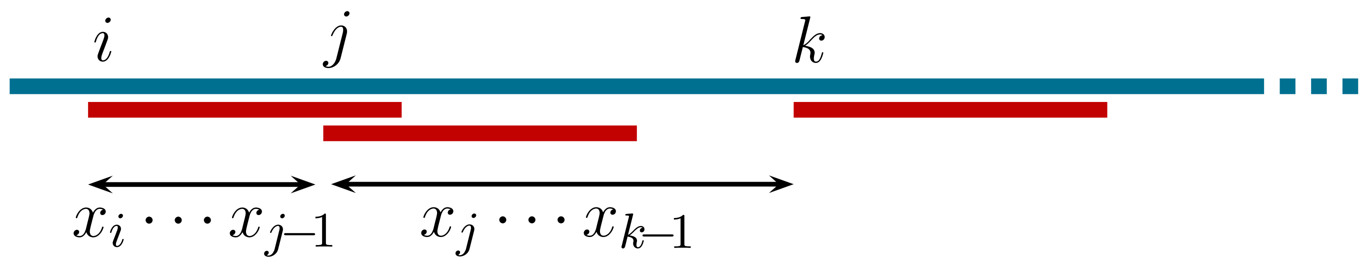

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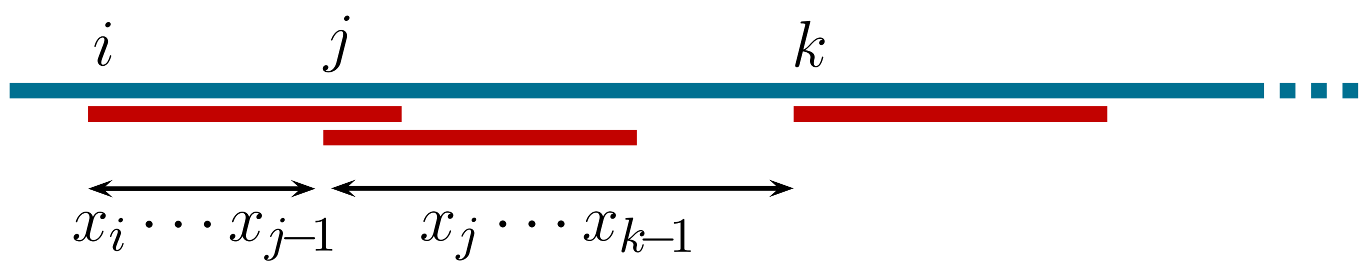

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is a *return word* of w in x.

- [Vuillon] An infinite binary word x is Sturmian *if and only if* each non empty factor of x has exactly two return words.

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Prop. Let $w \in \mathrm{Left}(x)$.
The two return words of w are

$$t_n \text{ and } t_n^i t_{n-1},$$

where n and i are determined by the length of w.

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- x' is written in a unique way as a concatenation of u, v:

$$x = p\, x' = p\ z_1 z_2 \cdots \quad (z_i \in \{u, v\}).$$

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- $x = 0c_\alpha$ with $\alpha = [0, \overline{3, 2}]$
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Δ

$$001 \mapsto 0$$

$$0010 \mapsto 1$$

$$D_w(x) = 0\;1\;0\;1\;0\;1\;0\;0\;1\;\cdots$$

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Θ: $0 \mapsto 001$, $1 \mapsto 0010$

$$D_w(x) = 0\,1\,0\,1\,0\,1\,0\,0\,1\ \cdots$$

- x' is obtained from this "*derivated word*" by applying the morphism $0 \mapsto 001$, $1 \mapsto 0010$ to it.

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Def. [Durand] The *derivated word* of x with respect to w is

$$D_w(x) = \Delta(z_1)\Delta(z_2) \cdots \; .$$

[$\longleftarrow$]

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Proposition. The derivated word $D_w(c_\alpha)$ of a characteristic Sturmian word c_α is a characteristic Sturmian word c_δ. The slope δ can be determined from α and w.

THEOREMS

An infinite word w is *substitutive* if it is the image by a literal morphism of a *morphic word*.
A morphism is *Sturmian* if the image of a Sturmian word is a Sturmian word.

Theorem. [Crisp *et al.*] [Araújo et Bruyère]
The following are equivalent

- α is a Sturm number;
- $D_w(c_\alpha) = c_\alpha$, for some $w \in \mathrm{Pref}(c_\alpha)$;
- c_α is a fixed point of a morphism.

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Theorem. [Araújo et Bruyère]
The following are equivalent

- α is a Sturm number;
- $D_w(x)$ has slope α, for some $w \in \mathrm{Left}(x)$;
- $\varphi(x)$ has slope α, for some Sturmian morphism φ;

for $x \in S_\alpha$.

THEOREMS (2)

Theorem. The following are equivalent

- the continued fraction of α is ultimately periodic;
- the set of all derivated words of c_α is finite;
- c_α is substitutive.

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- the continued fraction of α is ultimately periodic;
- there exist a Sturmian word y, and words x, x', such that $D_w(x)$, y and $D_{w'}(y)$ have the same slope;
- there exist a Sturmian word y, and two Sturmian morphisms ψ and φ, such that

  x and $\psi(y)$ have the same slope,

  y and $\varphi(y)$ have the same slope;

for $x \in S_\alpha$.

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Theorem. For $x \in R_\alpha$, the following are equivalent

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Question. Under which conditions is the set of derivated words of a Sturmian word finite?

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Dynamical systems and Sturmian words

- c_α the characteristic word
- $R_\alpha = \{\sigma^i(c_\alpha) \mid i \geqslant 1\}$ the suffixes of c_α
- S_α the closure of R_α

Words in S_α:

- same set of factors
- same return words with respect to a given factor w
- $\mathrm{Left}(x) = \mathrm{Pref}(c_\alpha)$