Restricted towers of Hanoi and morphisms

J.-P. Allouche and A. Sapir

The towers of Hanoi have been introduced by the French mathematician Lucas (under the pseudonym Claus) in 1883.

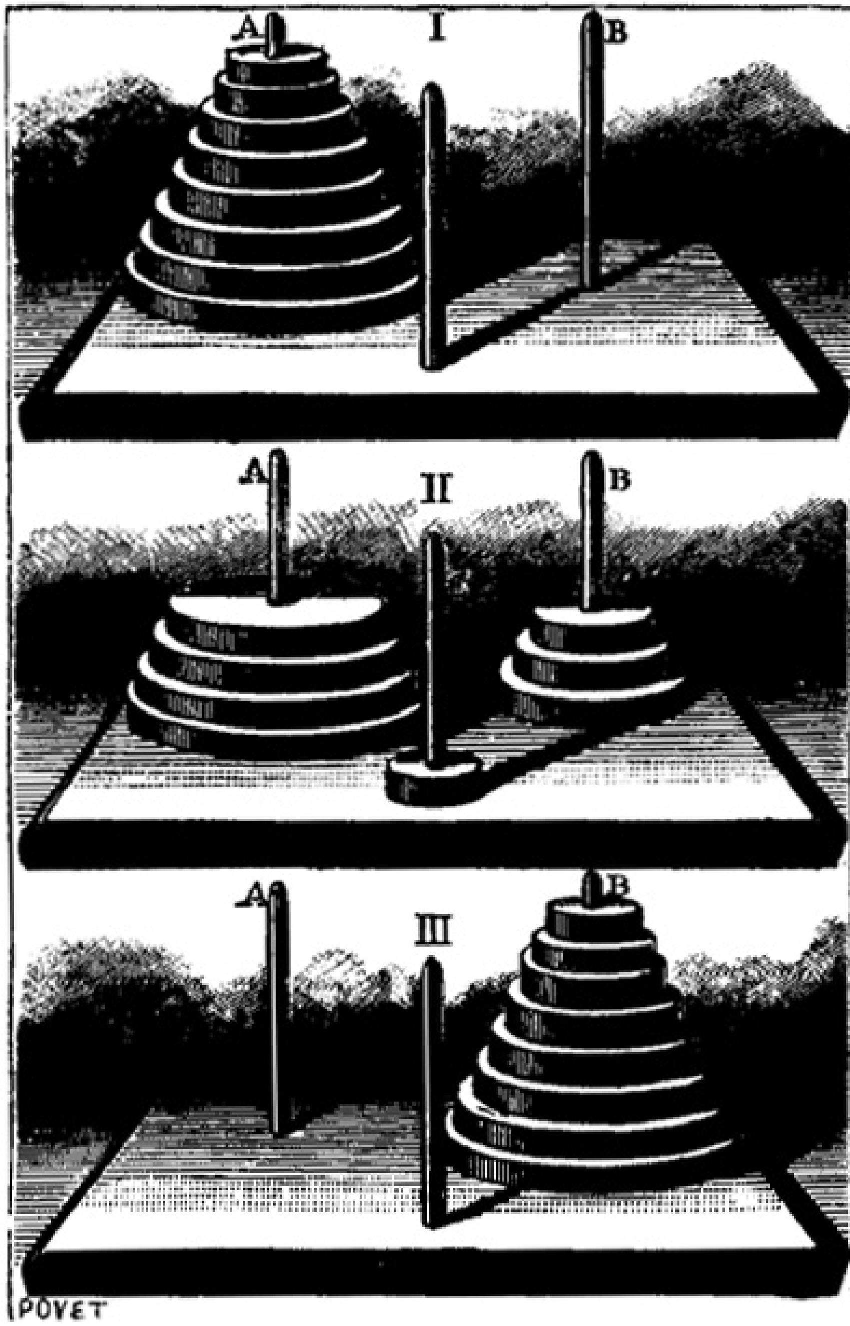
(Legend with monks and the end of the world.)

Three pegs, N disks of distinct diameters. At the beginning all disks are on a same peg.

The game consists of moving all disks to another peg where

– at each step the topmost disk on some peg is moved to the top of another peg,

– no disk can cover a smaller one.


A
I
B
A
II
B
A
III
B
POVET

Algorithms:

– recursive versus non-recursive (...)

– every other move takes the smallest disk on the next peg clockwise

Note:

– minimal number of moves $2^N - 1$

– memory

Free monoids

A finite set (alphabet), for example $\{0, 1\}$.

Define $\{0, 1\}^*$ to be the set of words (i.e., finite strings) on $\{0, 1\}$, including the empty word.

The set $\{0, 1\}^*$ equipped with the concatenation operation is a monoid ("the free monoid generated by $\{0, 1\}$").

$$011\ .\ 1010 := 0111010$$

What are the homomorphisms of (free) monoids?

Morphism (or substitution, or inflation rule)

Defined on letters.

$$0 \to 01, \;\; 1 \to 0$$

$$011 \to (01)(0)(0) = 0100$$

Uniform morphisms

$$0 \to 01, \;\; 1 \to 01$$

(this one has length 2)

Iterative fixed points of morphisms

Take the morphism $0 \to 01$, $1 \to 0$ and iterate it, starting from 0:

$$
\begin{array}{l}
0 \\
01 \\
010 \\
01001 \\
01001010 \\
0100101001001 \\
\ldots
\end{array}
$$

This gives an infinite sequence that is a fixed point of (the extension by continuity) of the morphism.

$$0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \ldots$$

This example is the (binary) Fibonacci sequence.

The morphism $0 \to 01, 1 \to 10$ gives rise to the Thue-Morse (Prouhet-Thue-Morse) sequence.

$$0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ \ldots$$

A *morphic sequence* is a pointwise image of an iterative fixed point of a morphism.

An *automatic sequence* is a pointwise image of an iterative fixed point of a uniform morphism.

Note:

– pointwise image = image by a uniform morphism of length 1,

– d-automatic := pointwise image of an iterative fixed point of a uniform morphism of length d.

Hanoi towers and automatic sequences

J.-P. A., F. Dress (1990)
J.-P. A., J. Bétréma, J. Shallit (1989)

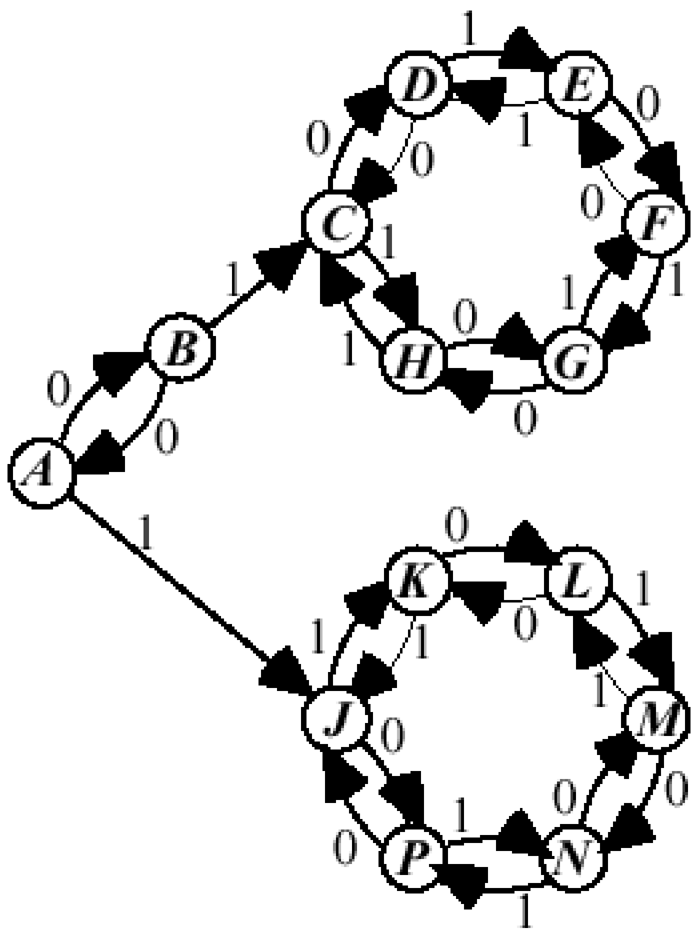
(i) There exists an infinite sequence on the six-letter alphabet of the possible moves whose prefixes of length $2^N - 1$ give the sequence of moves to transfer N disks from peg I to peg II if N is odd, and from peg I to peg III if N is even.

(ii) This infinite sequence is 2-automatic (actually it is the iterative fixed point of a uniform morphism of length 2).

The Hanoi sequence is the iterative fixed point of the morphism:

$$\begin{array}{lll} a \to a\overline{c}, & b \to c\overline{b}, & c \to b\overline{a} \\ \overline{a} \to ac, & \overline{b} \to cb, & \overline{c} \to ba \end{array}$$

on the alphabet of moves $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$



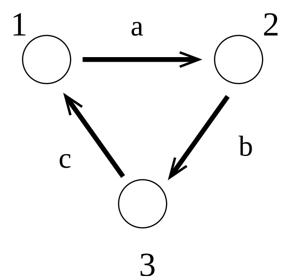
$$
\begin{array}{lllllllll}
\varphi(C) & = & \varphi(D) & = & \overline{c}, & \varphi(E) & = & \varphi(F) & = & \overline{a} \\
\varphi(G) & = & \varphi(H) & = & \overline{b}, & \varphi(J) & = & \varphi(P) & = & a \\
\varphi(K) & = & \varphi(L) & = & b, & \varphi(M) & = & \varphi(N) & = & c
\end{array}
$$

Remark:

– link between Hanoi towers and finite automata (morphisms)

– the kth move can be computed in time $\log k$ and bounded memory (compare with the classical algorithms)

Cyclic towers of Hanoi

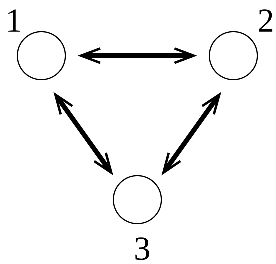


– introduced by M. D. Atkinson (1981)

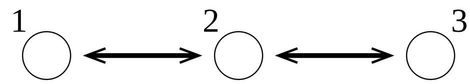
– an infinite morphism sequence gives the moves in an optimal algorithm (J.-P. A., F. Dress, 1990)

– this infinite sequence is **not** d-automatic for any d (J.-P. A., 1994)

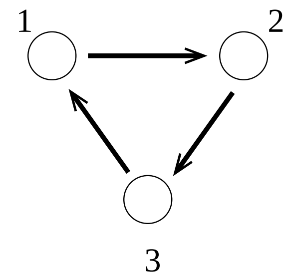
What are all possible Hanoi puzzles with restricted moves?



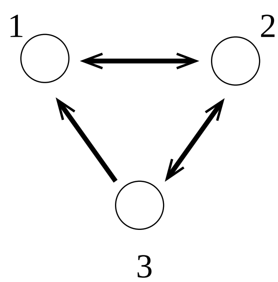
Complete



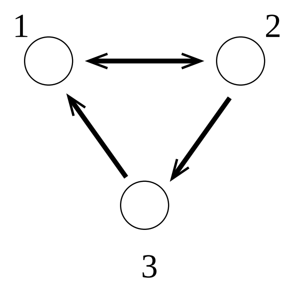
Three−in−a−row



Cyclic



Complete−−



Cyclic++

A. Sapir (2004) shows there are exactly 5 non-isomorphic Hanoi puzzles.

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He gives an optimal algorithm for each of these 5 cases.

He proves that the number of moves for transfering N disks has order λ^N, and he gives the 5 values of λ.

Looking at Sapir's algorithm permits to show that there exist corresponding infinite sequences of moves.

Furthermore these infinite sequences are *locally catenative*.

Locally catenative (example)

Define the sequence of words X_n by $X_0 := 0$, and for all $n \geq 1$, $X_{n+1} := X_n\overline{X_n}$, where $\overline{0} := 0$ and $\overline{1} := 0$. Then

$$\begin{aligned} X_0 &= 0 \\ X_1 &= 01 \\ X_2 &= 0110 \\ X_3 &= 01101001 \\ \ldots \end{aligned}$$

(did you recognize the limit sequence?)

Locally catenative $\approx$ morphic
(cf. J. Shallit, 1988)

Theorem (J.-P. A. and A. Sapir)

– The five infinite sequences of moves given by the optimal algorithms of Sapir are morphic.

– Furthermore the classical Hanoi sequence is the iterative fixed point of a uniform morphism of length 2, and the “three-in-a-row” Hanoi sequence is the iterative fixed point of a uniform morphism of length 3.

...

PERSPECTIVES

Morphic versus automatic?

An example. Take the celebrated Thue-Morse sequence

$$0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ \ldots$$

and count the number of 1's between consecutive 0's:

$$2\ 1\ 0\ 2\ 0\ 1\ 2\ 1\ \ldots$$

2 1 0 2 0 1 2 1 ...

J. Berstel proved (1978-1979) that this sequence is both the iterative fixed point of the morphism

$$2 \to 210, \ 1 \to 20, \ 0 \to 1$$

and the image under the map $x \to x \bmod 3$ of the uniform morphism

$$2 \to 21, \ 1 \to 02, \ 0 \to 04, \ 4 \to 20$$

on the alphabet $\{0, 1, 2, 4\}$.

How often does this phenomenon happen?

– Cobham's theorem (1969):
2- *and* 3-*automatic imply ultimately periodic*.

(More generally d- and d'-automatic, where d and d' are multiplicatively independent, i.e., $\log d/\log d' \notin \mathbb{Q}$.)

– Cobham's theorem

– Hansel's conjecture: idem with dominant eigenvalues of the incidence matrices (see partial results by F. Durand).

Note:

compute the dominant eigenvalue of the incidence matrix for the morphism

$$2 \to 210, \;\; 1 \to 20, \;\; 0 \to 1$$

What about the 5 Hanoi puzzles?

Two cases are still open (see partial result, see idea for cyclic Hanoi towers).

How to prove that a sequence is not automatic?