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# Adding Monotonic Counters to Automata and Transition Graphs

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Lehrstuhl für Informatik VII

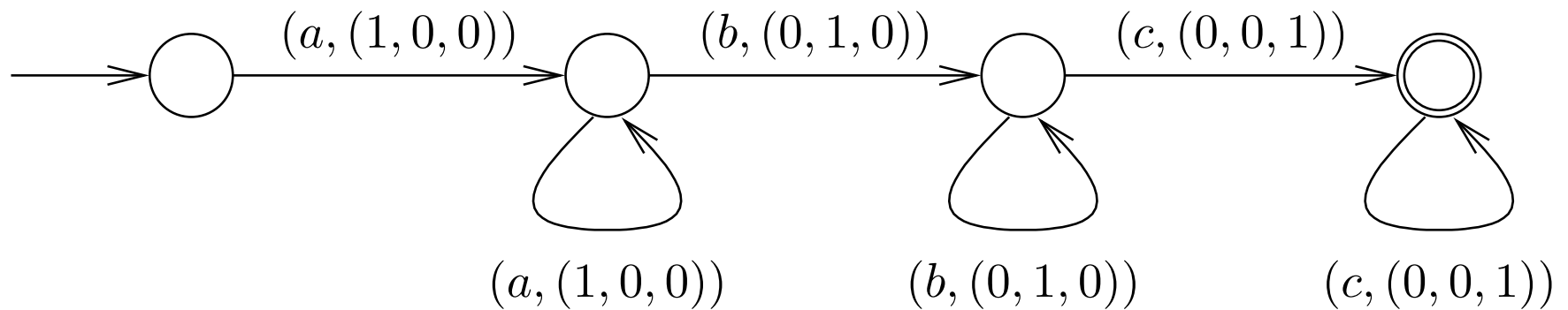


# Motivations: Parikh Automata (Klaedtke & Rueß, ICALP 2003)

Idea: how to recognize a language with *arithmetical properties*, such as

$$L_{abc} := \{a^k b^k c^k \mid k \geq 1\}?$$

- Use a finite automaton recognizing  $a^+ b^+ c^+$ .
- Assign a *vector* to each input symbol.
- Put a Presburger *constraint* on the summed vector  $(x, y, z)$ :  
 $x = y = z$ .



1. Parikh automata
  - Semi-linear sets and Parikh's theorem
  - Parikh automata and the Chomsky hierarchy
2. Monotonic-counter extensions of (infinite) graphs
  - Some classes of infinite graphs
  - Monotonic-counter extensions
3. Reachability problem

# Part 1

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## Parikh Automata

# Semi-Linear Sets and Parikh's Theorem

$A \subseteq \mathbb{N}^n$  *linear*:  $A = \{\bar{x}_0 + k_1\bar{x}_1 + \dots + k_m\bar{x}_m \mid k_1, \dots, k_m \in \mathbb{N}\}$   
for some  $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_m \in \mathbb{N}^n$

*Semi-linear set*: finite union of linear sets.

Example:  $B := \{(x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_1 < x_2 < x_3\}$  is linear:

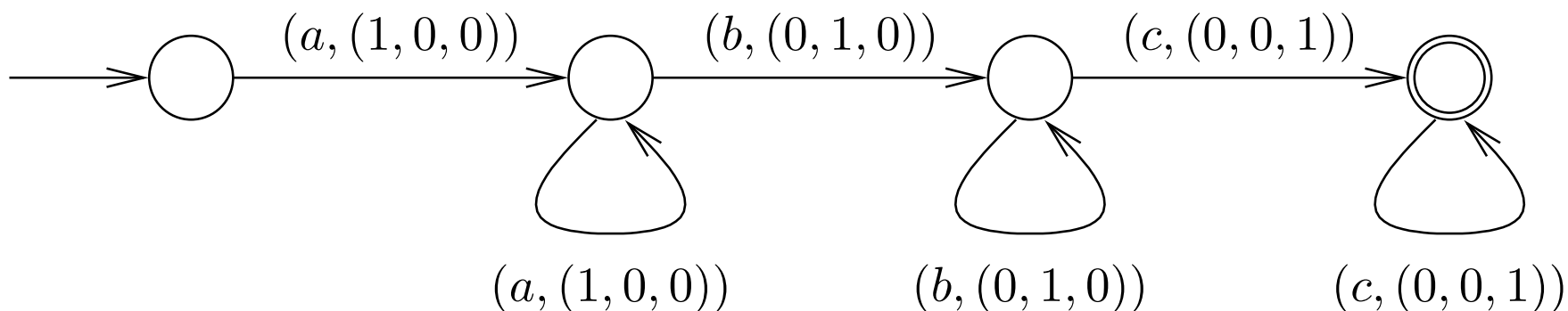
$$\{(0, 1, 2) + k_1(0, 0, 1) + k_2(0, 1, 1) + k_3(1, 1, 1) \mid k_1, k_2, k_3 \in \mathbb{N}\}.$$

Properties of semi-linear sets:

- effective closure under Boolean operations [Ginsburg & Spanier]
- equivalence to Presburger-definable sets [Ginsburg & Spanier]

*Parikh's theorem*: The *Parikh image* of any context-free language is effectively semi-linear.

# Parikh Automata



*Parikh finite automaton (Parikh-FA)*  $(\mathcal{A}, C)$  of dimension  $n \geq 1$  over  $\Sigma$ :

- finite automaton  $\mathcal{A}$  over  $\Sigma \times D$  ( $D \subseteq \mathbb{N}^n$  finite, nonempty)
- semi-linear set  $C \subseteq \mathbb{N}^n$

Word  $u := a_1 \cdots a_m$  is *accepted* iff

- $v := (a_1, \bar{d}_1) \cdots (a_m, \bar{d}_m) \in L(\mathcal{A})$  exists, for some  $\bar{d}_1, \dots, \bar{d}_m \in D$ ,
- *and*  $\Phi(v) := \bar{d}_1 + \cdots + \bar{d}_m \in C$ .

*extended Parikh mapping*  $\Phi: (\Sigma \times D)^* \rightarrow \mathbb{N}^n$

# Emptiness Problem

Lemma (Klaedtke & Rueß):

If the *Parikh image* of  $L \subseteq (\Sigma \times D)^*$  is effectively semi-linear, then also its *extended Parikh image*  $\Phi(L)$ .

Theorem (Klaedtke & Rueß):

The emptiness problem for Parikh-FA's is decidable.

*Proof idea.*

$$L(\mathcal{A}, C) \neq \emptyset \quad \text{iff} \quad \underbrace{\Phi(L(\mathcal{A})) \cap C \neq \emptyset}$$

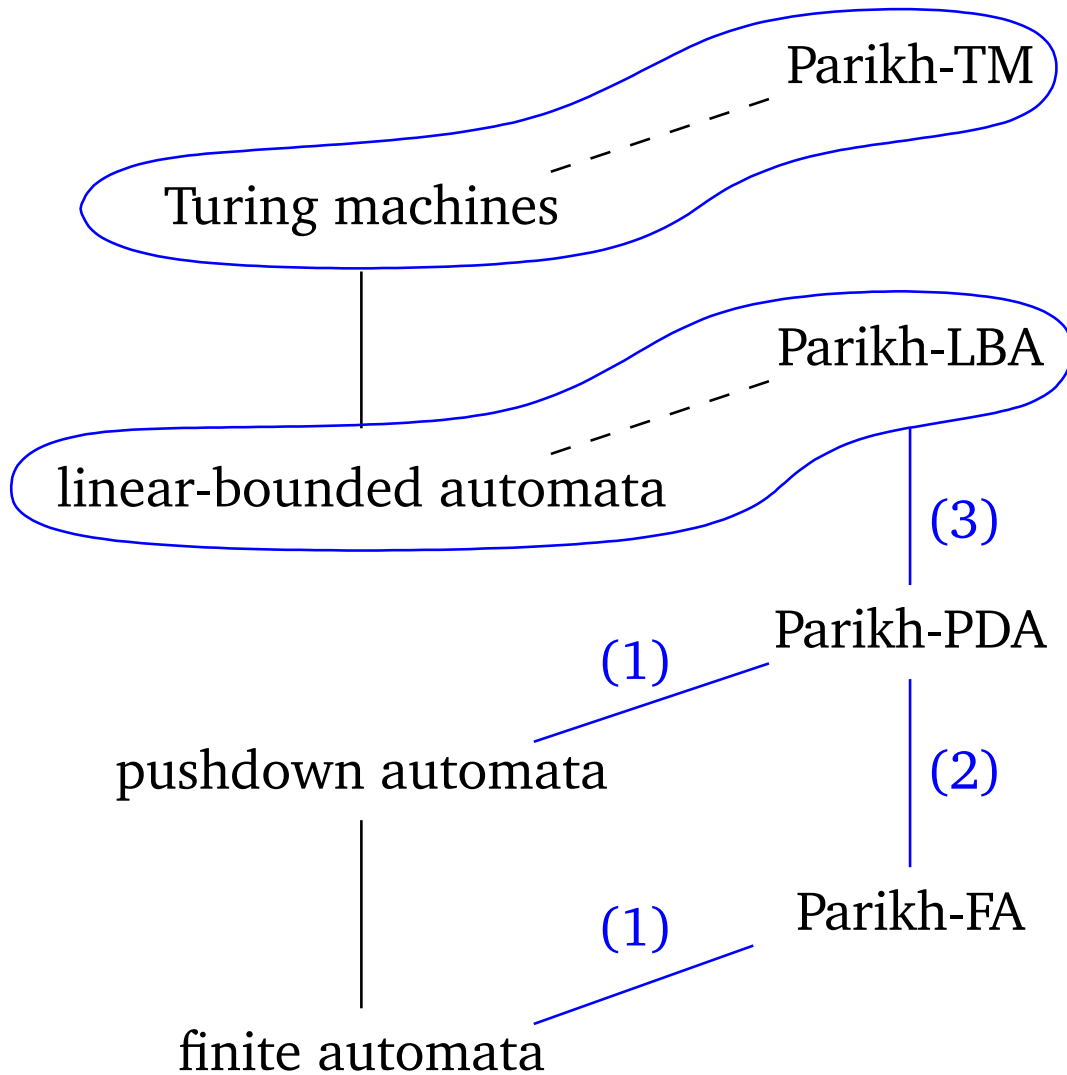
- Both sets are semi-linear.
- Intersection of semi-linear sets is effectively semi-linear.

$\implies$  (Non-)Emptiness is decidable.

□

# Parikh Automata and the Chomsky Hierarchy

Automata of the Chomsky hierarchy as the automaton component  $\mathcal{A}$ :



1.  $\{a^k b^k c^k \mid k \geq 1\}$
2.  $\{ww^R \mid w \in \{a, b\}^*\}$
3. semi-linearity of Parikh-PDA recognizable languages



## Part 2

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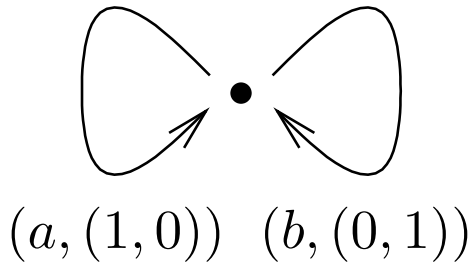
# Monotonic-Counter Extensions of (Infinite) Graphs

# Monotonic-Counter Graphs

$(\Sigma \times D)$ -labeled graph

$$G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D})$$

$(D \subseteq \mathbb{N}^n$  finite, nonempty)



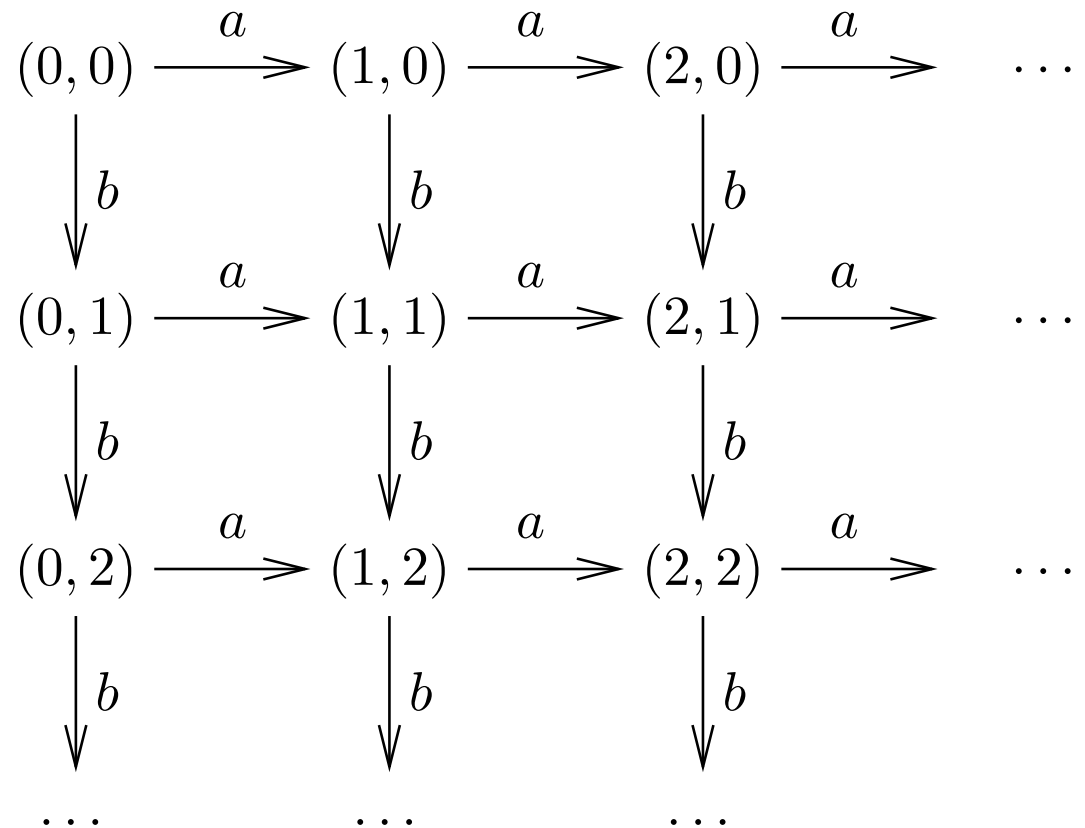
*Monotonic-counter extension of  $G$ :*

$\Sigma$ -labeled graph  $\tilde{G} := (\tilde{V}, (\tilde{E}_a)_{a \in \Sigma})$  with  $\tilde{V} := V \times \mathbb{N}^n$  and  $((\alpha, \bar{x}), (\beta, \bar{y})) \in \tilde{E}_a$  iff

$(\alpha, \beta) \in E_{(a,\bar{d})}$  and

$\bar{y} = \bar{x} + \bar{d}$ , for some  $\bar{d} \in D$ .

$\rightsquigarrow$



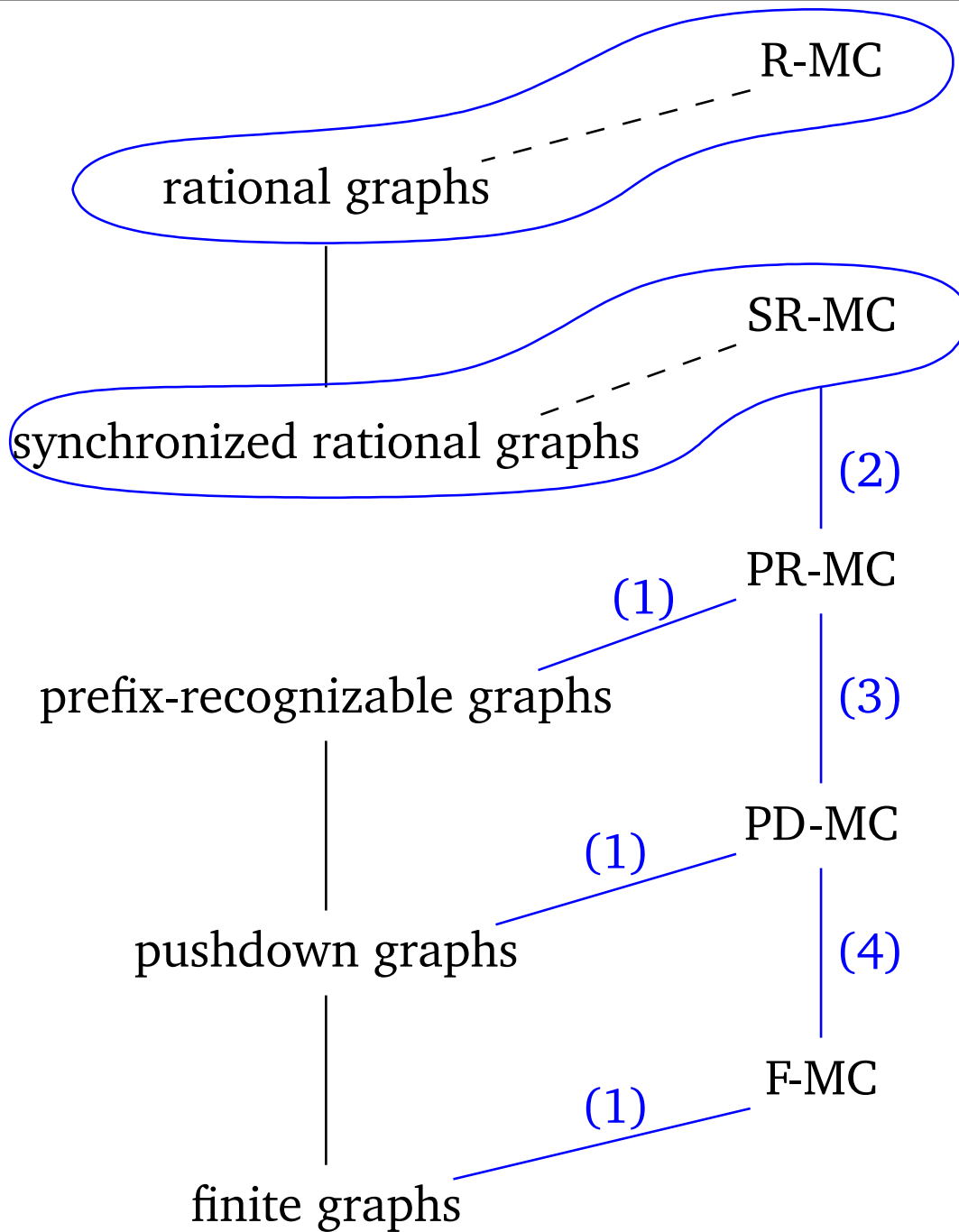
# Some Classes of Infinite Graphs with Finite Representations

Vertices: regular sets over alphabet  $\Gamma$

Edges: automaton-definable relations over words, e.g.:

- *pushdown graphs* [Muller & Schupp]: transitions of  $\varepsilon$ -free pushdown automata
- *prefix-recognizable graphs* [Caucal]: generalized prefix rewriting rules
- *synchronized rational graphs* or *automatic graphs* [Frougny & Sakarovitch, Blumensath & Grädel]: synchronized rational relations
- *rational graphs* [Morvan]: rational relations

# Hierarchy of Graph Classes



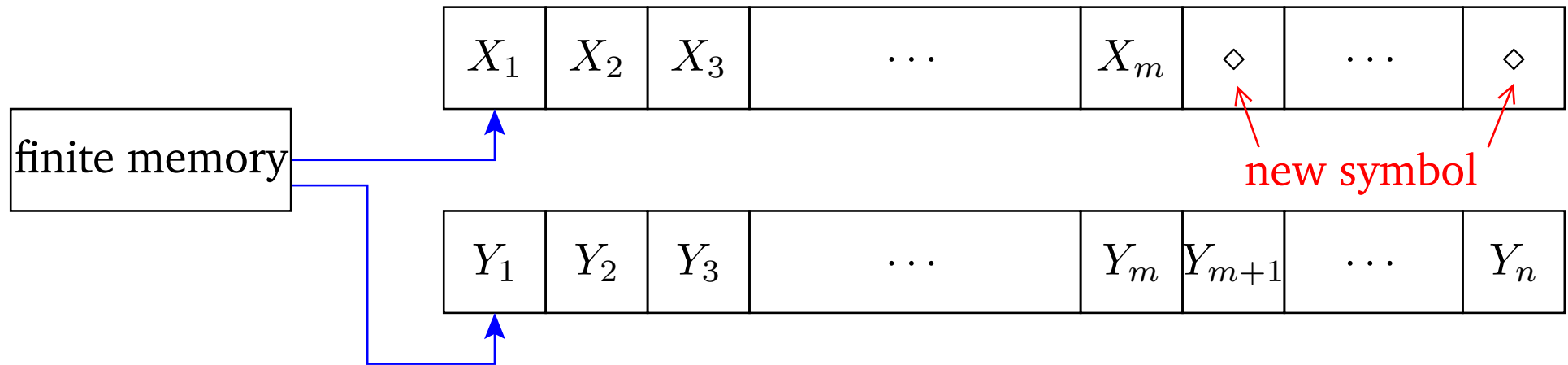
1. infinite two-dimensional grid
2. decidability of the reachability problem
3. graphs with vertices of unbounded degree
4. graphs with repetition-free cycles of unbounded length

# Synchronized Rational Graphs

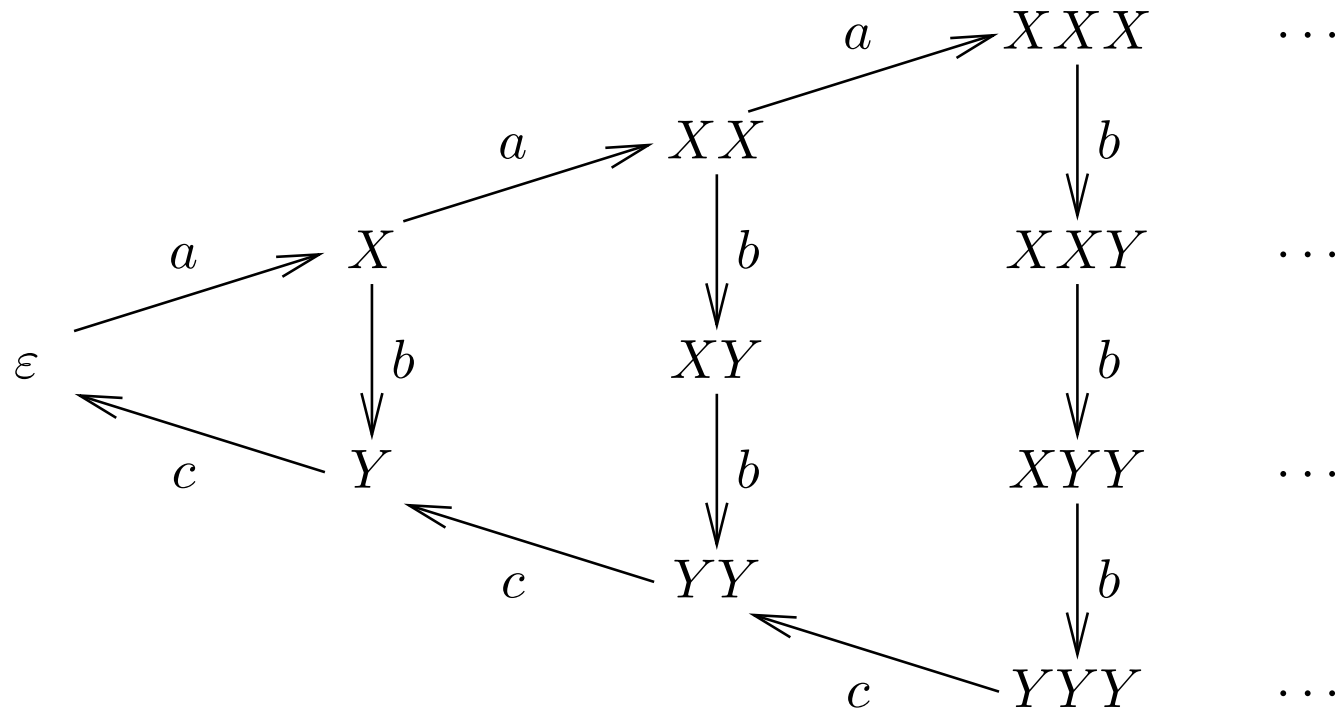
Vertices: regular set  $V$  over alphabet  $\Gamma$

Edges: synchronized rational relations, i.e. edge relation  $E_a$  is recognized by a *finite-state* automaton working on pairs

$(X_1 \cdots X_m, Y_1 \cdots Y_n) \in \Gamma^* \times \Gamma^*$  with two *one-way* input tapes and *simultaneously* moving input heads.



# Synchronized Rational Graphs: Example



# Monotonic-Counter Extensions of Synchronized Rational Graphs

$(\Sigma \times D)$ -labeled  
 synchronized rational graph  
 $G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D})$

$(D \subseteq \mathbb{N}^n$  finite, nonempty)

$\rightsquigarrow$   $\Sigma$ -labeled SRMC graph  
 $\tilde{G} := (\tilde{V}, (\tilde{E}_a)_{a \in \Sigma})$  with  $\tilde{V} := V \times \mathbb{N}^n$   
 and  $((\alpha, \bar{x}), (\beta, \bar{y})) \in \tilde{E}_a$  iff  
 $(\alpha, \beta) \in E_{(a,\bar{d})}$  and  
 $\bar{y} = \bar{x} + \bar{d}$ , for some  $\bar{d} \in D$ .

Proposition:  $\tilde{G}$  is synchronized rational.

*Proof sketch.* Encode vertex  $(\alpha, (x_1, \dots, x_n))$  of  $\tilde{G}$  by means of word

$$\underbrace{\#_1 \cdots \#_1}_{x_1} \cdots \underbrace{\#_n \cdots \#_n}_{x_n} \alpha$$

Define automaton for  $\tilde{E}_a$  working on pairs  $(\#_1^{x_1} \cdots \#_n^{x_n} \alpha, \#_1^{y_1} \cdots \#_n^{y_n} \beta)$ :

1. Guess a vector  $\bar{d} \in D$  and check whether  $\bar{x} + \bar{d} = \bar{y}$ .
2. Simulate the automaton for  $E_{(a,\bar{d})}$  on  $(\alpha, \beta)$ .

*Bounded delay* sufficient since  $D$  and  $\Gamma$  are finite. □

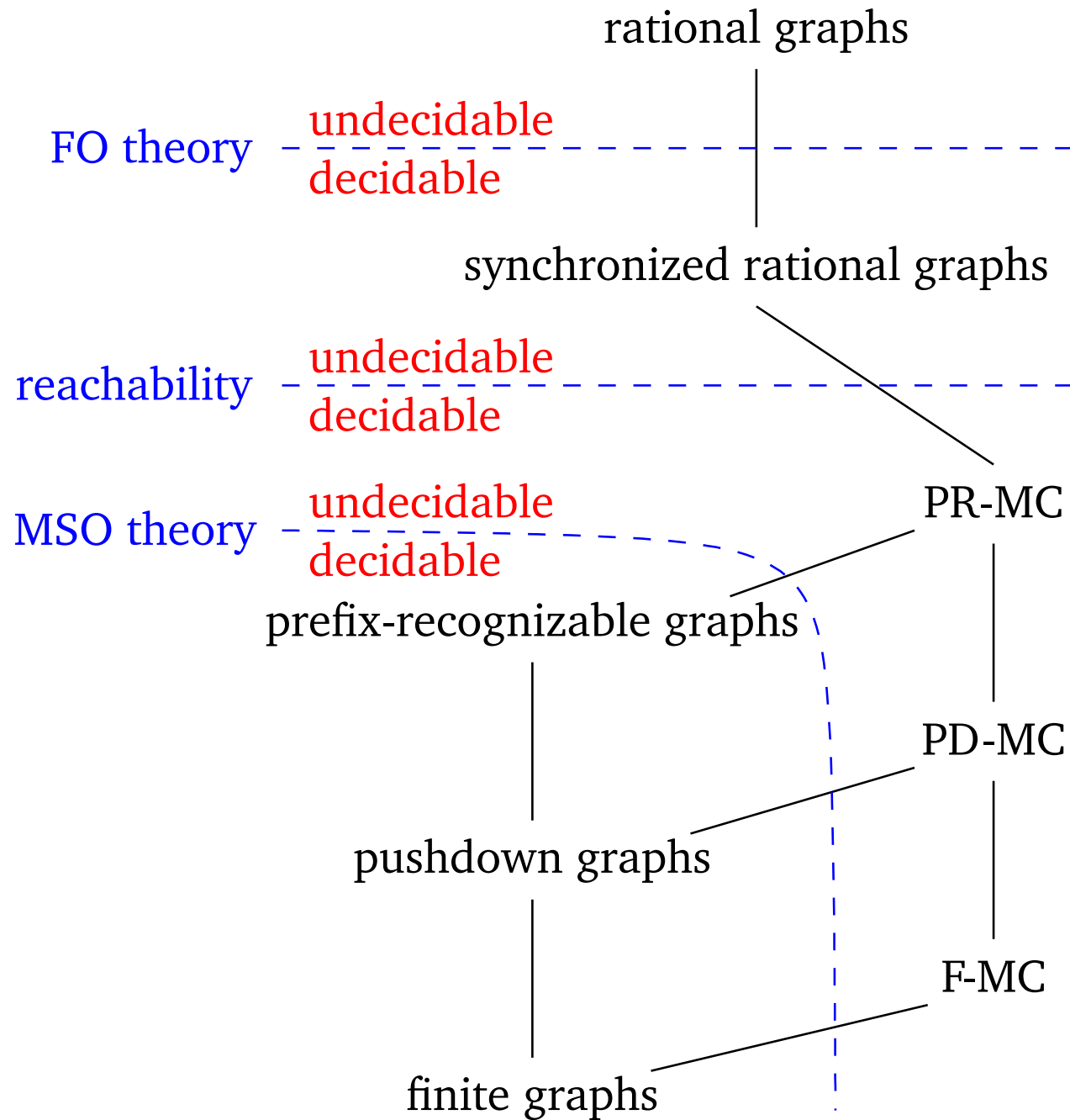
## Part 3

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# Reachability Problem



# Logical Decision Problems over Transition Graphs



# Prefix-Recognizable Graphs

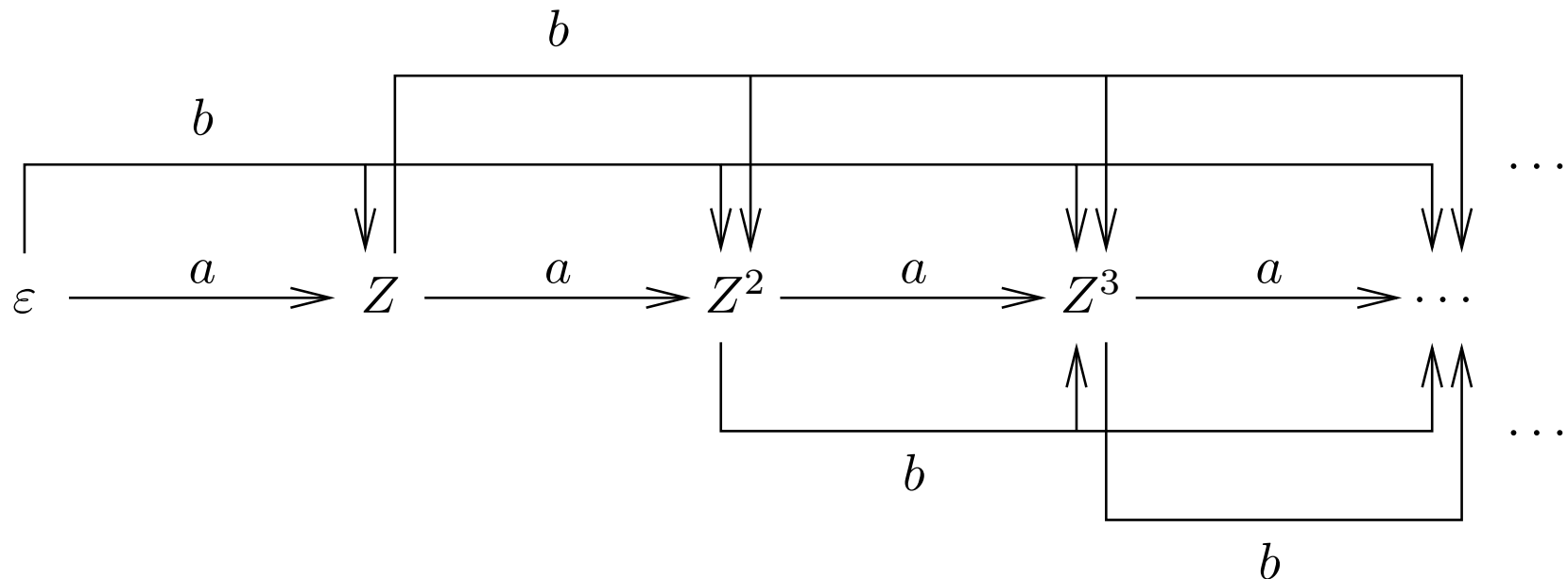
$$\Sigma := \{a, b\} \quad \Gamma := \{Z\}$$

*rule  $U \xrightarrow{a} V$  with  $U, V \subseteq \Gamma^*$  regular*

Prefix-rewriting system  $\mathfrak{R} := \{ \varepsilon \xrightarrow{a} Z, \varepsilon \xrightarrow{b} Z^+ \}$

Prefix-recognizable graph  $G = (V, E_a, E_b)$  defined by  $\mathfrak{R}$ :

- $V = \Gamma^*$ ,
- $E_a = \{(Z^i, Z^{i+1}) \mid i \in \mathbb{N}\}$ , and
- $E_b = \{(Z^i, Z^j) \mid i, j \in \mathbb{N} \text{ and } i < j\}$ .



# Monotonic-Counter Extensions of Prefix-Recognizable Graphs

$(\Sigma \times D)$ -labeled

prefix-recognizable graph

$$G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D}) \rightsquigarrow$$

$(D \subseteq \mathbb{N}^n$  finite, nonempty)

$\Sigma$ -labeled PRMC graph

$$\tilde{G} := (\tilde{V}, (\tilde{E}_a)_{a \in \Sigma}) \text{ with } \tilde{V} := V \times \mathbb{N}^n$$

and  $((\alpha, \bar{x}), (\beta, \bar{y})) \in \tilde{E}_a$  iff

$$(\alpha, \beta) \in E_{(a,\bar{d})} \text{ and}$$

$$\bar{y} = \bar{x} + \bar{d}, \text{ for some } \bar{d} \in D.$$

Reachability problem for  $\tilde{G}$ :

Given: *regular* sets  $U, U' \subseteq V$  of vertices in  $G$  and

*semi-linear* sets  $C, C' \subseteq \mathbb{N}^n$

Question: are there vertices  $(\alpha, \bar{x}) \in U \times C$  and

$(\beta, \bar{y}) \in U' \times C'$  in  $\tilde{G}$  such that

$(\beta, \bar{y})$  is reachable from  $(\alpha, \bar{x})$ ?

# Reachability Problem for PRMC Graphs

**Proposition:** The reachability problem for the monotonic-counter extension of any prefix-recognizable graph is decidable.

*Proof sketch.* Let  $L \subseteq (\Sigma \times D)^*$  be the *traces* of  $G$  with  $U$  and  $U'$  as the set of initial and final vertices, respectively.

Show

$$U' \times C' \text{ is reachable from } U \times C \quad \text{iff} \quad (C + \Phi(L)) \cap C' \neq \emptyset$$

Right-hand side:

- $L$  is context-free and effectively constructible (Caucal 2003).
  - $\Phi(L)$  is effectively semi-linear.
  - Effective closure of semi-linear sets under  $+$  and  $\cap$ .
- $\implies$  (Non-)Emptiness is decidable



# Summary

- Parikh automata correspond to adding monotonic counters with an additional semi-linearity test.
- No increase in language recognition power for linear-bounded automata, in contrast to finite and pushdown automata.
- Application to transition graphs: no increase for synchronized rational graphs, but for pushdown and prefix-recognizable graphs.
- For prefix-recognizable graphs, reachability remains decidable.

# Further Prospects

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- Using more general frameworks than monotonic counters, e.g. reversal-bounded counters.
- Allowing intermediate tests during computations.
- Comparing Parikh automata to reversal-bounded counter automata.
- Using more general arithmetical conditions than semi-linear sets.